Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 1$

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Abstract: In this paper we prove that the defocusing, quintic nonlinear Schrödinger initial value problem is globally well-posed and scattering for $u_0 \in L^2(\mathbf{R})$. To do this, we will prove a frequency localized interaction Morawetz estimate similar to the estimate made in [11]. Since we are considering an L^2 - critical initial value problem we will localize to low frequencies.

1 Introduction

The quintic nonlinear Schrödinger initial value problem is given by

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0,x) &= u_0 \in L^2(\mathbf{R}), \end{aligned} \tag{1.1}$$

where $F(u) = \mu|u|^4 u$, $\mu = \pm 1$, $u(t) : \mathbf{R} \to \mathbf{C}$. When $\mu = +1$ (1.1) is said to be defocusing and when $\mu = -1$ (1.1) is said to be focusing. It was observed in [4] that the solution to (1.1) conserves mass,

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{1.2}$$

and energy

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \frac{\mu}{6}\int |u(t,x)|^6 dx = E(u(0)). \tag{1.3}$$

The initial value problem (1.1) also obeys a scaling symmetry. If $u(t,x)$ is a solution to (1.1) on a time interval $[0,T]$, then

$$u_\lambda(t,x) = \frac{1}{\lambda^{1/2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \tag{1.4}$$

is a solution to (1.1) on $[0, \lambda^2 T]$ with $u(0,x) = \frac{1}{\lambda^{1/2}} u_0(\frac{x}{\lambda})$.

$$\|\frac{1}{\lambda^{1/2}}u_0(\frac{x}{\lambda})\|_{L^2(\mathbf{R})} = \|u_0(x)\|_{L^2(\mathbf{R})}. \tag{1.5}$$

Therefore, (1.1) is called L^2 - critical or mass critical.

A solution to (1.1) obeys Duhamel's formula

Definition 1.1 *$u : I \times \mathbf{R}^d \to \mathbf{C}$, $I \subset \mathbf{R}$ is a solution to (1.1) if for any compact $J \subset I$, $u \in C_t^0 L_x^2(J \times \mathbf{R}^d) \cap L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)$, and for all $t, t_0 \in I$,*

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^{t} e^{i(t-\tau)\Delta}F(u(\tau))d\tau. \tag{1.6}$$

The space $L_{t,x}^6(J \times \mathbf{R})$ arises from the Strichartz estimates. This norm is also invariant under the scaling (1.4).

Definition 1.2 *A solution to (1.1) defined on $I \subset \mathbf{R}$ blows up forward in time if there exists $t_0 \in I$ such that*

$$\int_{t_0}^{\sup(I)}\int |u(t,x)|^6 dxdt = \infty. \tag{1.7}$$

u blows up backward in time if there exists $t_0 \in I$ such that

$$\int_{\inf(I)}^{t_0}\int |u(t,x)|^6 dxdt = \infty. \tag{1.8}$$

Definition 1.3 *A solution $u(t,x)$ to (1.1) is said to scatter forward in time if there exists $u_+ \in L^2(\mathbf{R}^d)$ such that*

$$\lim_{t\to\infty} \|e^{it\Delta}u_+ - u(t,x)\|_{L^2(\mathbf{R}^d)} = 0. \tag{1.9}$$

A solution is said to scatter backward in time if there exists $u_- \in L^2(\mathbf{R}^d)$ such that

$$\lim_{t\to-\infty} \|e^{it\Delta}u_- - u(t,x)\|_{L^2(\mathbf{R}^d)} = 0. \tag{1.10}$$

Theorem 1.1 *If $\|u_0\|_{L^2(\mathbf{R})}$ is sufficiently small, then (1.1) is globally well-posed and scatters to a free solution as $t \to \pm\infty$.*

Proof: See [4], [5]. $\square$

We will recall the proof of this theorem in §2. [4], [5] also proved that (1.1) is locally well-posed for $u_0 \in L^2(\mathbf{R})$ on some interval $[0, T]$, where $T(u_0) > 0$ depends on the profile of the initial data, not just it size $\|u_0\|_{L^2(\mathbf{R})}$.

Theorem 1.2 *Given $u_0 \in L^2(\mathbf{R}^2)$ and $t_0 \in \mathbf{R}$, there exists a maximal lifespan solution u to (1.1) defined on $I \subset \mathbf{R}$ with $u(t_0) = u_0$. Moreover,*

1. *I is an open neighborhood of t_0.*
2. *If $\sup(I)$ or $\inf(I)$ is finite, then u blows up in the corresponding time direction.*
3. *The map that takes initial data to the corresponding solution is uniformly continuous on compact time intervals for bounded sets of initial data.*
4. *If $\sup(I) = \infty$ and u does not blow up forward in time, then u scatters forward to a free solution. If $\inf(I) = -\infty$ and u does not blow up backward in time, then u scatters backward to a free solution.*

Proof: See [4], [5]. $\square$

There are known counterexamples to (1.1) globally well-posed and scattering in the focusing case, $\mu = -1$. There are no known counterexamples in the defocusing case. Therefore, it has been conjectured

Conjecture 1.3 *For $d \geq 1$, the defocusing, mass critical nonlinear Schrödinger initial value problem (1.1) is globally well-posed for $u_0 \in L^2(\mathbf{R}^d)$ and all solutions scatter to a free solution as $t \to \pm\infty$.*

This conjecture has already been verified for $d \geq 2$.

Theorem 1.4 *When $d = 2$, (1.1) is globally well-posed and scattering for $u_0 \in L^2(\mathbf{R}^2)$.*

Proof: See [23] for a proof in the radial case, [17] for a proof in the non-radial case.

Theorem 1.5 *When $d \geq 3$, (1.1) is globally well-posed and scattering for $u_0 \in L^2(\mathbf{R}^d)$.*

Proof: See [24], [32] for a proof in the radial case, [18] for a proof in the nonradial case.

In this paper we tackle the case $d = 1$ and prove

Theorem 1.6 *(1.1) is globally well-posed and scattering for $u_0 \in L^2(\mathbf{R})$, $\mu = +1$.*

This completes the proof of the conjecture in the defocusing case.

Remark: [23] and [24] also proved global well-posedness and scattering for the focusing, mass-critical initial value problem

$$\begin{aligned} iu_t + \Delta u &= -|u|^{4/d}u, \\ u(0,x) &= u_0, \end{aligned} \tag{1.11}$$

with radial data and mass less than the mass of the ground state when $d \geq 2$. Much of the analysis in this paper carries over directly to the focusing case. Therefore, whenever possible we will prove theorems without regard for the sign of μ.

Outline of the Proof. In this paper we use the concentration compactness method, which is a modification of the induction on energy method. The induction on energy method was introduced in [3] to prove global well-posedness and scattering for the defocusing energy-critical initial value problem in $\mathbf{R}^3$ for radial data.

[23], [24], [32], [18], and [17] used the concentration compactness method. Since (1.1) is globally well-posed for small $\|u_0\|_{L^2(\mathbf{R})}$, if (1.1), $\mu = +1$ is not globally well-posed for all $u_0 \in L^2(\mathbf{R})$, then there must be a minimum $\|u_0\|_{L^2(\mathbf{R})} = m_0$ where global well-posedness fails. [34] showed that for conjecture 1.3 to fail, there must exist a minimal mass blowup solution with a number of additional properties.

Theorem 1.7 *Suppose conjecture 1.3 fails when $d = 1$. Then there exists a maximal lifespan solution on $I \subset \mathbf{R}$, $[0,\infty) \subset I$, $\|u(t)\|_{L^2_x(\mathbf{R}^d)} = m_0$ which is almost periodic modulo scaling and blows up both forward and backward in time. Moreover, $N(t) \leq 1$ for $t \in [0,\infty)$, $N(0) = 1$, and*

$$\int_0^\infty \int |u(t,x)|^6 dx = \infty. \tag{1.12}$$

Additionally, there exists a set $K \subset L^2(\mathbf{R})$, K is precompact in $L^2(\mathbf{R}^d)$ such that for all $t \in I$ there exists $Q_t \in K$, $x(t), \xi(t) : I \to \mathbf{R}$ with

$$u(t,x) = \frac{1}{N(t)^{1/2}} e^{ix\cdot\xi(t)} Q_t(\frac{x - x(t)}{N(t)}). \tag{1.13}$$

Proof: See [23], [34], and section four of [32].

Remark: This is also true for a minimal mass blowup solution to the focusing problem (1.1), $\mu = -1$.

We will then consider two subcases separately,

$$\int_0^\infty N(t)^3 dt < \infty, \tag{1.14}$$

and

$$\int_0^\infty N(t)^3 dt = \infty. \tag{1.15}$$

We will exclude (1.14) by proving additional regularity, which prevents $N(t) \searrow 0$ as $t \to \infty$. For (1.15) we will not prove any additional regularity. Instead, we will rely on a frequency localized interaction Morawetz estimate. (See [11] for such an estimate in the energy-critical case.) Since we are truncating to low frequencies, our method is very similar to the almost Morawetz estimates that are often used in conjunction with the I-method. (See [1], [8], [9], [10], [12], [6], [19], [15], [13], and [14] for more information on the I-method.)

2 Function Spaces and linear estimates

Linear Strichartz Estimates:

Definition 2.1 *A pair (p, q) will be called an admissible pair for $d = 1$ if $\frac{2}{p} = (\frac{1}{2} - \frac{1}{q})$, and $p \geq 4$.*

Theorem 2.1 *If $u(t, x)$ solves the initial value problem*

$$\begin{aligned} iu_t + \Delta u &= F(t), \\ u(0, x) &= u_0, \end{aligned} \tag{2.1}$$

on an interval I, then

$$\|u\|_{L_t^p L_x^q(I \times \mathbf{R})} \lesssim_{p,q,\tilde{p},\tilde{q}} \|u_0\|_{L^2(\mathbf{R})} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I \times \mathbf{R})}, \tag{2.2}$$

for all admissible pairs (p, q), $(\tilde{p}, \tilde{q})$. $\tilde{p}'$ denotes the Lebesgue dual of $\tilde{p}$.

Proof: See [31].

(2.2) motivates the definition of the Strichartz space.

Definition 2.2 *Define the norm*

$$\|u\|_{S^0(I \times \mathbf{R})} \equiv \sup_{(p,q) \text{ admissible}} \|u\|_{L_t^p L_x^q(I \times \mathbf{R})}. \tag{2.3}$$

$$S^0(I \times \mathbf{R}) = \{u : \|u\|_{S^0(I \times \mathbf{R})} < \infty\}. \tag{2.4}$$

We also define the space $N^0(I \times \mathbf{R})$ to be the space dual to $S^0(I \times \mathbf{R})$ with appropriate norm. Then in fact,

$$\|u\|_{S^0(I\times\mathbf{R})} \lesssim \|u_0\|_{L^2(\mathbf{R})} + \|F\|_{N^0(I\times\mathbf{R})}. \tag{2.5}$$

Theorem 2.2 *(1.1) is globally well-posed when $\|u_0\|_{L^2(\mathbf{R})}$ is small.*

Proof: By (2.5) and the definition of S^0, N^0,

$$\begin{aligned}\|u\|_{S^0((-\infty,\infty)\times\mathbf{R})} &\lesssim \|u_0\|_{L^2(\mathbf{R})} + \|u\|^5_{L^6_{t,x}((-\infty,\infty)\times\mathbf{R})} \\ &\lesssim \|u_0\|_{L^2(\mathbf{R})} + \|u\|^5_{S^0((-\infty,\infty)\times\mathbf{R})}.\end{aligned} \tag{2.6}$$

By the continuity method, if $\|u_0\|_{L^2(\mathbf{R})}$ is sufficiently small, then we have global well-posedness. We can also obtain scattering with this argument. □

Now define the function

$$A(m) = \sup\{\|u\|_{S^0((-\infty,\infty)\times\mathbf{R}^2)} : \text{ u solves (1.1)}, \|u(0)\|_{L^2(\mathbf{R}^2)} = m\}. \tag{2.7}$$

If we can prove $A(m) < \infty$ for any m, then we have proved global well-posedness and scattering.

Using a stability lemma from [34] we can prove that $A(m)$ is an upper semicontinuous function of m, which proves that $\{m : A(m) = \infty\}$ is a closed set. This implies that if global well-posedness and scattering does not hold in the defocusing case for all $u_0 \in L^2(\mathbf{R})$, then there must be a minimum m_0 with $A(m_0) = \infty$. We will discuss the properties of a minimal mass blowup solution more in the next section.

We will also need the Littlewood-Paley decomposition at various points throughout the paper. Let $\phi \in C_0^\infty(\mathbf{R})$, radial, $0 \leq \phi \leq 1$,

$$\phi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| > 2. \end{cases} \tag{2.8}$$

Then define the frequency truncation

$$\mathcal{F}(P_{\leq N}u) = \phi(\frac{\xi}{N})\hat{u}(\xi). \tag{2.9}$$

Let $P_{>N}u = u - P_{\leq N}u$ and $P_N u = P_{\leq 2N}u - P_{\leq N}u$. We will also depart from the customary notation and say

$$P_{1/2}u = P_{\leq 1}u. \tag{2.10}$$

Throughout the paper it will be necessary to make a Littlewood-Paley decomposition with $\xi_0 \neq 0$ at the origin. Let

$$\tilde{P}_{N,\xi_0} u = e^{ix\cdot\xi_0} P_N(e^{-ix\cdot\xi_0} u). \tag{2.11}$$

Function Spaces
We utilize the function spaces which are a superposition of free solutions to the Schrodinger equation. See [26], [21] for more information.

Definition 2.3 *Let $1 \leq p < \infty$. Then U_{Δ}^p is an atomic space, where atoms are piecewise solutions to the linear equation.*

$$u = \sum_k 1_{[t_k, t_{k+1})} e^{it\Delta} u_k, \qquad \sum_k \|u_k\|_{L^2}^p = 1. \tag{2.12}$$

For any function u,

$$\|u\|_{U_{\Delta}^p} = \inf\{\sum_{\lambda} |c_{\lambda}| : u = \sum_{\lambda} c_{\lambda} u_{\lambda}, u_{\lambda} \text{ are } U_{\Delta}^p \text{ atoms}\} \tag{2.13}$$

For any $1 \leq p < \infty$, $U_{\Delta}^p \subset L^{\infty} L^2$. Additionally, U_{Δ}^p functions are continuous except at countably many points and right continuous everywhere.

Definition 2.4 *Let $1 \leq p < \infty$. Then V_{Δ}^p is the space of right continuous functions $u \in L^{\infty}(L^2)$ such that*

$$\|v\|_{V_{\Delta}^p}^p = \|v\|_{L^{\infty}(L^2)}^p + \sup_{\{t_k\} \nearrow} \sum_k \|e^{-it_k\Delta} v(t_k) - e^{-it_{k+1}\Delta} v(t_{k+1})\|_{L^2}^p. \tag{2.14}$$

The supremum is taken over increasing sequences t_k.

Theorem 2.3 *The function spaces U_{Δ}^p, V_{Δ}^q obey the embeddings*

$$U_{\Delta}^p \subset V_{\Delta}^p \subset U_{\Delta}^q \subset L^{\infty}(L^2), \qquad p < q. \tag{2.15}$$

Let DU_{Δ}^p be the space of functions

$$DU_{\Delta}^p = \{(i\partial_t + \Delta) u; u \in U_{\Delta}^p\}. \tag{2.16}$$

There is the easy estimate

$$\|u\|_{U_{\Delta}^p} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t + \partial_x^2) u\|_{DU_{\Delta}^p}. \tag{2.17}$$

Finally, there is the duality relation

$$(DU_\Delta^p)^* = V_\Delta^{p'}. \tag{2.18}$$

These spaces are also closed under truncation in time.

$$\begin{aligned} \chi_I : U_\Delta^p \to U_\Delta^p, \\ \chi_I : V_\Delta^p \to V_\Delta^p. \end{aligned} \tag{2.19}$$

Proof: See [21]. □

Lemma 2.4 *Suppose $J = I_1 \cup I_2$, $I_1 = [a, b]$, $I_2 = [b, c]$, $a \leq b \leq c$.*

$$\begin{aligned} \|u\|_{U_\Delta^p(J \times \mathbf{R})}^p \leq \|u\|_{U_\Delta^p(I_1 \times \mathbf{R})}^p + \|u\|_{U_\Delta^p(I_2 \times \mathbf{R})}^p \\ \|u\|_{U_\Delta^p(I_1 \times \mathbf{R})} \leq \|u\|_{U_\Delta^p(J \times \mathbf{R}^d)}. \end{aligned} \tag{2.20}$$

Proof: See [17].

Proposition 2.5

$$\|P_N((e^{it\Delta} u_0)(e^{-it\Delta} v_0))\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R})} \lesssim \frac{1}{N^{1/2}} \|u_0\|_{L^2(\mathbf{R})} \|v_0\|_{L^2(\mathbf{R})}. \tag{2.21}$$

If the supports of $\hat{u}_0(\xi)$ and $\hat{v}_0(\xi)$ are separated by distance N,

$$\|(e^{it\Delta} u_0)(e^{it\Delta} v_0)\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R})} \lesssim \frac{1}{N^{1/2}} \|u_0\|_{L^2(\mathbf{R})} \|v_0\|_{L^2(\mathbf{R})}. \tag{2.22}$$

Proof: We prove (2.21).

$$\tilde{G}(\tau, \xi) = \int e^{-it\tau} \int_{\xi = \eta_1 + \eta_2} e^{-it\eta_1^2} e^{it\eta_2^2} \hat{u}_0(\eta_1) \hat{v}_0(\eta_2) d\eta_1 dt$$

$$= \int_{\xi = \eta_1 + \eta_2} \delta(\tau + \eta_1^2 - \eta_2^2) \hat{u}_0(\eta_1) \hat{v}_0(\eta_2) d\eta_1.$$

Take $\tilde{F}(\tau, \xi)$ with $\|\tilde{F}(\tau, \xi)\|_{L_{\tau,\xi}^2(\mathbf{R} \times \mathbf{R})} = 1$, F supported on $|\xi| \sim N$.

$$\int \int \tilde{F}(-\tau, -\xi) \tilde{G}(\tau, \xi) d\tau d\xi = \int \int \tilde{F}((\eta_1 + \eta_2)(\eta_1 - \eta_2), \eta_1 + \eta_2) \hat{u}_0(\eta_1) \hat{v}_0(\eta_2) d\eta_1 d\eta_2.$$

Making a change of variables, this proves (2.21). (2.22) can be proved in a similar fashion. □

Proposition 2.6 *Suppose $\hat{u}_0$ is supported on $|\xi| \sim N_1$ and $\hat{v}_0$ is supported on $|\xi| \sim N_2$, $N_1 << N_2$. Then*

$$\|(e^{\pm it\Delta} u_0)(e^{\pm it\Delta} v_0)\|_{L^3_{t,x}(\mathbf{R} \times \mathbf{R})} \lesssim (\frac{N_1}{N_2})^{1/4} \|u_0\|_{L^2(\mathbf{R})} \|v_0\|_{L^2(\mathbf{R})}. \tag{2.23}$$

Proof: By proposition 2.5,

$$\|(e^{\pm it\Delta} u_0)(e^{\pm it\Delta} v_0)\|_{L^2_{t,x}(\mathbf{R} \times \mathbf{R})} \lesssim (\frac{1}{N_2})^{1/2} \|u_0\|_{L^2(\mathbf{R})} \|v_0\|_{L^2(\mathbf{R})}.$$

Also, combining Strichartz estimates and the Sobolev embedding theorem,

$$\|(e^{\pm it\Delta} u_0)(e^{\pm it\Delta} v_0)\|_{L^6_{t,x}(\mathbf{R} \times \mathbf{R})} \lesssim \|e^{\pm it\Delta} u_0\|_{L^\infty_{t,x}(\mathbf{R} \times \mathbf{R})} \|e^{\pm it\Delta} u_0\|_{L^6_{t,x}(\mathbf{R} \times \mathbf{R})} \lesssim N_1^{1/2} \|u_0\|_{L^2(\mathbf{R})} \|v_0\|_{L^2(\mathbf{R})}.$$

The proposition follows by interpolation. □

Right now, we know that our minimal mass blowup solution is concentrated in both space and frequency, that is,

$$\int_{|x - x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t,x)|^2 dx < \eta, \tag{2.24}$$

$$\int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\hat{u}(t,\xi)|^2 d\xi < \eta. \tag{2.25}$$

Since we will be using the interaction Morawetz estimate, we will not need to track the movement of $x(t)$, however, it will be very important to track the movement of $\xi(t)$. One weapon to partially counter the movement of $\xi(t)$ is the Galilean transformation.

Theorem 2.7 *Suppose $u(t,x)$ solves*

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0,x) &= u_0. \end{aligned} \tag{2.26}$$

Then $v(t,x) = e^{-it|\xi_0|^2} e^{ix \cdot \xi_0} u(t, x - 2\xi_0 t)$ solves the initial value problem

$$\begin{aligned} iv_t + \Delta v &= F(v), \\ v(0,x) &= e^{ix \cdot \xi_0} u(0,x). \end{aligned} \tag{2.27}$$

Proof: This follows by direct calculation. $\square$

If $u(t,x)$ obeys (2.24) and (2.25) and $v(t,x) = e^{-it|\xi_0|^2} e^{ix\cdot\xi_0} u(t, x - 2\xi_0 t)$, then

$$\int_{|\xi - \xi_0 - \xi(t)| \geq C(\eta) N(t)} |\hat{v}(t,\xi)|^2 d\xi < \eta, \tag{2.28}$$

$$\int_{|x - 2\xi_0 t - x(t)| \geq \frac{C(\eta)}{N(t)}} |v(t,x)|^2 dx < \eta. \tag{2.29}$$

Remark: This will be useful to us later because it shifts $\xi(t)$ by a fixed amount $\xi_0 \in \mathbf{R}^d$. For example, this allows us to set $\xi(0) = 0$.

Lemma 2.8 *If J is an interval with*

$$\|u\|_{L^6_{t,x}(J \times \mathbf{R})} \leq C, \tag{2.30}$$

then for $t_1, t_2 \in J$,

$$N(t_1) \sim_{C, m_0} N(t_2). \tag{2.31}$$

Proof: See [24], [23], or [33]. $\square$

Now if $\|u\|_{L^6_{t,x}([0,T] \times \mathbf{R}^d)} \leq C$, partition $[0,T]$ into $\sim \frac{C^6}{\epsilon_0^6}$ subintervals and iterate. $\square$

We can control the movement of $\xi(t)$ with a similar argument.

Lemma 2.9 *Partition $J = [0, T_0]$ into subintervals $J = \cup J_k$ such that*

$$\|u\|_{L^6_{t,x}(J_k \times \mathbf{R}^d)} \leq \epsilon_0. \tag{2.32}$$

Let $N(J_k) = \sup_{t \in J_k} N(t)$. Then

$$|\xi(0) - \xi(T_0)| \lesssim \sum_k N(J_k), \tag{2.33}$$

which is the sum over the intervals J_k.

Proof: Again take $\eta = \frac{m_0^2}{1000}$. Let $t_1, t_2 \in J_k$. By Strichartz estimates,

$$\| \int_{t_1}^{t} e^{i(t-\tau)\Delta} |u(\tau)|^4 u(\tau) d\tau \|_{L^2_x(\mathbf{R})} \leq \frac{m_0}{1000}. \tag{2.34}$$

By (2.24) and (2.25)

$$\int_{|\xi-\xi(t_1)|\geq C(\frac{m_0^2}{1000})N(t_1)} |\hat{u}(t_1,\xi)|^2 d\xi \leq \frac{m_0^2}{1000}, \tag{2.35}$$

and

$$\int_{|\xi-\xi(t_2)|\geq C(\frac{m_0^2}{1000})N(t_2)} |\hat{u}(t_2,\xi)|^2 d\xi \leq \frac{m_0^2}{1000}. \tag{2.36}$$

By Duhamel's formula, conservation of mass, (2.34), (2.35), and (2.36), the balls $|\xi - \xi(t)| \leq C(\frac{m_0^2}{1000})N(t_1)$, $|\xi - \xi(t)| \leq C(\frac{m_0^2}{1000})N(t_2)$ must intersect, $|\xi(t_1) - \xi(t_2)| \leq 3C(\frac{m_0^2}{1000})(N(t_1) + N(t_2))$. By the triangle inequality and lemma 2.8,

$$|\xi(T_0) - \xi(0)| \leq \sum_k |\xi(t_k) - \xi(t_{k+1})| \lesssim \sum_k N(t_k). \tag{2.37}$$

□

Next, we quote a result,

Lemma 2.10 *If $u(t,x)$ is a minimal mass blowup solution on an interval J,*

$$\int_J N(t)^2 dt \lesssim \|u\|_{L_{t,x}^6(J\times\mathbf{R})}^6 \lesssim 1 + \int_J N(t)^2 dt. \tag{2.38}$$

Proof: See [24].

Finally we will prove a lemma that will be useful to us when analyzing the blowup scenarios with $N(t) \leq 1$.

Lemma 2.11 *Suppose u is a minimal mass blowup solution with $N(t) \leq 1$. Suppose also that J is some interval partitioned into subintervals J_k with $\|u\|_{L_{t,x}^6(J_k\times\mathbf{R})} = \epsilon_0$ on each J_k. Again let*

$$N(J_k) = \sup_{J_k} N(t). \tag{2.39}$$

Then,

$$\sum_{J_k} N(J_k) \sim \int_J N(t)^3 dt. \tag{2.40}$$

Proof: By lemma 2.10,

$$\int_J N(t)^2 \lesssim \|u\|_{L_{t,x}^6(J\times\mathbf{R})}^6. \tag{2.41}$$

Since $\|u\|_{L^6_{t,x}(J_k \times \mathbf{R})} = \epsilon_0$, by (2.38),

$$\int_{J_k} N(t)^3 dt \lesssim N(J_k) \int_{J_k} N(t)^2 \lesssim \epsilon_0^6 N(J_k),$$

so

$$\int_J N(t)^3 dt \lesssim \sum_{J_k} N(J_k).$$

On the other hand, by the Duhamel formula,

$$\|u\|_{L^4_t L^\infty_x (J_k \times \mathbf{R})} \lesssim \|u_0\|_{L^2(\mathbf{R})} + \|u\|^5_{L^6_{t,x}(J_k \times \mathbf{R})} \lesssim 1. \tag{2.42}$$

Interpolating this with

$$\|u_{|\xi - \xi(t)| \geq C(\eta) N(t)}\|_{L^\infty_t L^2_x (J_k \times \mathbf{R})} \leq \eta^{1/2}, \tag{2.43}$$

we have

$$\|u_{|\xi - \xi(t)| \geq C(\eta(\epsilon)) N(t)}\|_{L^6_{t,x}(J_k \times \mathbf{R})} \leq \frac{\epsilon_0}{1000}, \tag{2.44}$$

for a small, fixed $\eta(\epsilon_0) > 0$. By the Sobolev embedding theorem,

$$\|u_{|\xi - \xi(t)| \leq C(\eta(\epsilon_0)) N(t)}(t)\|_{L^6_x(\mathbf{R}^2)} \lesssim [C(\eta(\epsilon_0)) N(t)]^{\frac{1}{3}}. \tag{2.45}$$

Therefore,

$$\epsilon_0^6 \lesssim \int_{J_k} C(\eta(\epsilon_0))^2 N(t)^2 dt.$$

Since $N(t_1) \sim N(t_2)$ for $t_1, t_2 \in J_k$, this implies

$$N(J_k) \lesssim \int_{J_k} N(t)^3 dt. \tag{2.46}$$

Summing up over subintervals proves the lemma. $\Box$

3 A norm adapted to $\xi(t)$, $N(t)$ constant

As a warm-up, we will treat the minimal mass blowup scenario $N(t) \equiv 1$, $\xi(t) \equiv 0$, $\mu = \mp 1$. Rescaling,

$$u_\lambda(t,x) = \frac{1}{\lambda^{1/2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}), \tag{3.1}$$

$N(t) \equiv \frac{1}{\lambda}$. We will choose to treat the case $N(t) = \delta$, $\delta > 0$ sufficiently small so that $\delta < \epsilon^{10}$, and dropping the λ from $u_\lambda(t)$,

$$\|P_{> \frac{\delta^{1/2}}{32}} u(t)\|_{L_t^\infty L_x^2((-\infty,\infty)\times \mathbf{R}^2)} < \epsilon, \tag{3.2}$$

and for any $a \in \mathbf{R}$,

$$\|u\|_{L_t^4 L_x^\infty([a,a+1]\times \mathbf{R}^2)} + \|u\|_{L_{t,x}^6([a,a+1]\times \mathbf{R}^2)} \leq \epsilon_0. \tag{3.3}$$

The semi-norm we are about to define is adapted to the case $N(t) \equiv \delta$. This semi-norm will be generalized in the next section to treat the case when $N(t)$ and $\xi(t)$ are free to move around.

Definition 3.1 *Let N_j be a dyadic integer.*

$$\begin{aligned} \|u\|_{X_{N_j}^k}^2 = \sum_{1 \leq N_i \leq N_j} \frac{N_i}{N_j} \sum_{l=0}^{\frac{N_j}{N_i}-1} \|P_{N_i} u\|_{U_\Delta^2([kN_j + lN_i, kN_j + (l+1)N_i]\times \mathbf{R})}^2 \\ + \sum_{N_j < N_i} \|P_{N_i} u\|_{U_\Delta^2([kN_j,(k+1)N_j]\times \mathbf{R})}^2. \end{aligned} \tag{3.4}$$

Now let M be some dyadic integer,

$$\|u\|_{X_M([0,M]\times \mathbf{R})}^2 \equiv \sup_{1 \leq N_j \leq M} \sup_{0 \leq k \leq \frac{M}{N_j}} \|u\|_{X_{N_j}^k}^2. \tag{3.5}$$

Similarly, we can define

$$\|u\|_{X_M([a,a+M]\times \mathbf{R})}^2 \tag{3.6}$$

for any $a \in \mathbf{R}$.

Remark: $\|u\|_{X_M([0,M]\times \mathbf{R})}$ is only a semi-norm since if $f(t)$ is a nonzero function supported on $|\xi| < 1$, $\|f(t)\|_{X_M([0,M]\times \mathbf{R})} \equiv 0$. Therefore, we need to say something about a minimal mass blowup solution at low frequencies.

Lemma 3.1 *Suppose $u(t)$ is a minimal mass blowup solution to* (1.1)*, $\mu = \pm 1$, and J is an interval with*

$$\|u\|_{L_{t,x}^6(J\times \mathbf{R})} + \|u\|_{L_t^4 L_x^\infty(J\times \mathbf{R})} \leq \epsilon_0, \tag{3.7}$$

and $N(t) = \delta$ on J. Then

$$\|P_{>\delta^{1/2}} u(t)\|_{U_\Delta^2(J\times \mathbf{R})} \lesssim \epsilon. \tag{3.8}$$

Proof: Let $J = [a, b]$. By Duhamel's formula, for $t \in J$,

$$u(t) = e^{i(t-a)\Delta} u(a) - i \int_a^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau. \tag{3.9}$$

Since

$$\|P_{> \frac{\delta^{1/2}}{32}} u(t)\|_{L_t^\infty L_x^2(J \times \mathbf{R})} \leq \epsilon,$$

$$\|P_{> \frac{\delta^{1/2}}{32}} e^{i(t-a)\Delta} u(a)\|_{U_\Delta^2(J \times \mathbf{R})} \lesssim \epsilon.$$

Also,

$$\| \int_a^t e^{i(t-\tau)\Delta} P_{>\delta^{1/2}} (|u(\tau)|^4 u(\tau)) d\tau \|_{U_\Delta^2(J \times \mathbf{R})} \lesssim \|P_{>\delta^{1/2}} (|u(\tau)|^4 u(\tau))\|_{L_t^{4/3} L_x^1(J \times \mathbf{R})}$$

$$\lesssim \|P_{> \frac{\delta^{1/2}}{32}} u\|_{L_t^\infty L_x^2(J \times \mathbf{R})} \|u\|^4_{L_t^{16/3} L_x^4(J \times \mathbf{R})} \lesssim \epsilon_0^4 \epsilon. \square$$

Remark: Using the exact same arguments, if J is an interval with

$$\|u\|_{L_{t,x}^6(J \times \mathbf{R})} + \|u\|_{L_t^4 L_x^\infty(J \times \mathbf{R})} \leq \epsilon_0, \tag{3.10}$$

$$\|P_{>N(J)\delta^{1/2}} u\|_{U_\Delta^2(J \times \mathbf{R})} \lesssim \epsilon. \tag{3.11}$$

Theorem 3.2 *Suppose $u(t)$ is a minimal mass blowup solution to*

$$iu_t + \Delta u = F(u). \tag{3.12}$$

There exists a fixed constant C such that for $\epsilon, \delta(\epsilon) > 0$ sufficiently small,

$$\|u\|_{X_M([0,M] \times \mathbf{R})} \leq C\epsilon \tag{3.13}$$

for all dyadic M, $1 \leq M < \infty$.

Sketch of Proof: Theorem 3.2 is proved by induction. By lemma 3.1,

$$\|P_1 u(t)\|_{U_\Delta^2([a,a+1] \times \mathbf{R})} \leq C\epsilon. \tag{3.14}$$

Suppose that for any dyadic integer M, $1 \leq M < \infty$,

$$\|u\|_{X_M([a,a+M] \times \mathbf{R})} \leq \frac{C}{2}\epsilon + \frac{C}{2}(\epsilon^2 + \|u\|^2_{X_M([a,a+M] \times \mathbf{R}^2)}), \tag{3.15}$$

C is independent of $\epsilon > 0$. Then we are able to prove theorem 3.2 by induction. Suppose that for $M \leq N$,

$$\|u(t)\|_{X_M([a,a+M]\times\mathbf{R})} \leq C\epsilon, \tag{3.16}$$

for a fixed constant C, $\epsilon > 0$, and for any $a \in \mathbf{R}$. Then making a crude estimate,

$$\|u(t)\|_{X_{2N}([a,a+2N]\times\mathbf{R})} \leq 2C\epsilon. \tag{3.17}$$

By (3.15), (3.17),

$$\|u(t)\|_{X_{2N}([a,a+2N]\times\mathbf{R})} \leq \frac{C}{2}\epsilon + \frac{C}{2}(\epsilon^2 + (2C\epsilon)^2). \tag{3.18}$$

For $\epsilon > 0$ sufficiently small, this implies that for $a \in \mathbf{R}$,

$$\|u(t)\|_{X_{2N}([a,a+2N]\times\mathbf{R})} \leq C\epsilon, \tag{3.19}$$

closing the induction. □

The proof of an estimate of the form (3.15) will occupy the bulk of the paper. In fact, we will prove an estimate of the form (3.15) for a generalization of $\|u\|_{X_M([a,a+M]\times\mathbf{R})}$ used to treat the case when $N(t)$ need not be constant. (3.15) will be a special case of the more general result.

The purpose of this section is to discuss the simpler case in the hopes that the main idea is more evident, since it is not obscured by the technical details that arise when $\xi(t)$ and $N(t)$ are free to move around.

4 Estimates when $N(t)$, $\xi(t)$ are free to vary

In this section we will generalize the seminorm in the previous section to adapt it to the case when $N(t)$ and $\xi(t)$ are free to vary. We will define the seminorm $\tilde{X}_M([0,T])$ on the time interval $[0,T]$ to be an analogue of the $X_M([0,M])$ norm defined in the previous section.

Suppose $[0,T] = \cup_{l=1}^{M} J_l$, with $\|u\|_{L^6_{t,x}(J_l\times\mathbf{R})} = \epsilon_0$, $\sum_{J_l} N(J_l) = \delta M$. We will call the individual J_l subintervals the small intervals. We want to partition $[0,T]$ at level N_i for $1 \leq N_i \leq M$. If $N(J_l) > \frac{\delta N_i}{2}$ then we will call J_l a red interval at level N_i.

A union $G = \cup J_l$ of N_i consecutive small intervals with

$$\sum_{J_l \subset G} N(J_l) \leq \delta N_i$$

and $N(J_l) \leq \frac{\delta N_i}{2}$ for each $J_l \subset G$ will be called a length green interval at level N_i. A union G of $\leq N_i$ consecutive small intervals J_l with

$$\frac{\delta N_i}{2} < \sum_{J_l \subset G} N(J_l) \leq \delta N_i$$

will be called a weight green interval at level N_i.

A union Y of $< N_i$ consecutive small intervals with

$$\sum_{J_l \subset Y} N(J_l) \leq \frac{\delta N_i}{2}$$

will be called a yellow interval at level N_i.

$[0, T]$ will be partitioned so that every yellow interval Y lies immediately to the left of a red interval, or that $T \in Y$. If there is a yellow interval Y, and the small interval J_l to the right of Y satisfies $N(J_l) \leq \frac{\delta N_i}{2}$ then we take $Y \cup J_l = Y^*$. Y^* is the union of $\leq N_i$ small intervals with

$$\sum_{J_l \subset Y^*} N(J_l) \leq \delta N_i.$$

If Y^* is the union of N_i small intervals then Y^* is a length green interval. If

$$\frac{\delta N_i}{2} < \sum_{J_l \subset Y^*} N(J_l) \leq \delta N_i,$$

then Y^* is a weight green interval. If

$$\sum_{J_l \subset Y^*} N(J_l) \leq \frac{\delta N_i}{2}$$

and Y^* is the union of $< N_i$ small intervals, then Y^* remains a yellow interval. If $T \notin Y^*$ and the small interval to the right of Y^* is not red, then repeat the above procedure.

Remark: We will always say that $[0, T]$ is a green interval at level M.

Remark: The reader should think of the yellow intervals at level N_i as the scraps left over after carving out the red and green intervals at level N_i.

We also want to apply the seminorms in the previous section to the case when $\xi(t)$ is free to travel around in $\mathbf{R}$. This seminorm was defined in [17] for any dimension d, $d \geq 1$.

Definition 4.1 *For a green interval $G_\alpha^{N_i} = [a,b]$, let $\xi(G_\alpha^{N_i}) = \xi(a)$. $\xi(Y_\alpha^{N_i})$ and $\xi(R_\alpha^{N_i})$ can be defined in a similar manner.*

If $G_k^{N_j}$ is a green interval at level N_j, then

$$\|u\|^2_{X(G_k^{N_j})} \equiv \sum_{1 \leq N_i \leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i} \cap G_k^{N_j} \neq \emptyset} \|\tilde{P}_{\xi(G_\alpha^{N_i}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u\|^2_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R}^d)}$$
$$+ \sum_{N_j < N_i} \|\tilde{P}_{\xi(G_k^{N_j}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u\|^2_{U^2_\Delta(G_k^{N_j} \times \mathbf{R}^d)} + \sup_{1 \leq N_i < N_j} \sup_{Y_\alpha^{N_i} \cap G_k^{N_j} \neq \emptyset} \|\tilde{P}_{\xi(Y_\alpha^{N_i}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u\|^2_{U^2_\Delta(Y_\alpha^{N_i} \times \mathbf{R}^d)}. \tag{4.1}$$

For a yellow interval at level N_j,

$$\|u\|^2_{X(Y_k^{N_j})} \equiv \sum_{1 \leq N_i \leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i} \cap Y_k^{N_j} \neq \emptyset} \|P_{\xi(G_\alpha^{N_i}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u\|^2_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R}^d)}$$
$$+ \sum_{N_j \leq N_i} \|P_{\xi(Y_k^{N_j}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u\|^2_{U^2_\Delta(Y_k^{N_j} \times \mathbf{R}^d)} + \sup_{1 \leq N_i < N_j} \sup_{Y_\alpha^{N_i} \cap Y_k^{N_j} \neq \emptyset} \|P_{\xi(Y_\alpha^{N_i}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u\|^2_{U^2_\Delta(Y_\alpha^{N_i} \times \mathbf{R}^d)}. \tag{4.2}$$

Then,

$$\|u\|^2_{\tilde{X}_M([0,T] \times \mathbf{R}^d)} \equiv \sup_{1 \leq N_j \leq M} \sup_{G_k^{N_j} \subset [0,T]} \|u\|^2_{X(G_k^{N_j})} + \sup_{1 \leq N_j \leq M} \sup_{Y_k^{N_j} \subset [0,T]} \|u\|^2_{X(Y_k^{N_j})}. \tag{4.3}$$

Also for a dyadic integer N, $1 \leq N \leq M$ define the norm

$$\|u\|^2_{\tilde{X}_N([0,T] \times \mathbf{R}^d)} \equiv \sup_{1 \leq N_j \leq N} \sup_{G_k^{N_j} \subset [0,T]} \|u\|^2_{X(G_k^{N_j})} + \sup_{1 \leq N_j \leq N} \sup_{Y_k^{N_j} \subset [0,T]} \|u\|^2_{X(Y_k^{N_j})}. \tag{4.4}$$

We first prove than an estimate on $\|u\|_{X(G_k^{N_j})}$ gives control over $\|P_{\xi(G_k^{N_j}, N_i} u\|_{U^2_\Delta(G_k^{N_j} \times \mathbf{R})}$ for a dyadic frequency N_i, along with Strichartz estimates of $P_{\xi(G_k^{N_j}), N_i} u$.

Lemma 4.1 *For a dyadic frequency $1 \leq N_j$,*

$$\sum_{1 \leq N_i \leq N_j} (\frac{N_i}{N_j})[\sum_{G_\alpha^{N_i} \cap G_k^{N_j} \neq \emptyset} \|P_{\xi(G_\alpha^{N_i}), N_i} u\|^2_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R}^d)} + \sum_{Y_{\alpha'}^{N_i} \cap G_k^{N_j} \neq \emptyset} \|P_{\xi(Y_{\alpha'}^{N_i}), N_i} u\|^2_{U^2_\Delta(Y_{\alpha'}^{N_i} \times \mathbf{R}^d)}$$
$$+ \sum_{R_{\alpha''}^{N_i} \subset G_k^{N_j}} \|P_{\xi(R_{\alpha''}^{N_i}), N_i} u\|^2_{U^2_\Delta(R_{\alpha''}^{N_i} \times \mathbf{R}^d)}] + \sum_{N_j < N_i} \|P_{\xi(G_k^{N_j}), N_i} u\|^2_{U^2_\Delta(G_k^{N_j} \times \mathbf{R}^d)} \lesssim \|u\|^2_{X(G_k^{N_j})} + \epsilon^2. \tag{4.5}$$

Similarly,

$$\sum_{1 \leq N_i \leq N_j} (\frac{N_i}{N_j})[\sum_{G_{\alpha}^{N_i} \cap Y_k^{N_j} \neq \emptyset} \|P_{\xi(G_{\alpha}^{N_i}), N_i} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R}^d)}^2 + \sum_{Y_{\alpha'}^{N_i} \cap Y_k^{N_j} \neq \emptyset} \|P_{\xi(Y_{\alpha'}^{N_i}), N_i} u\|_{U_{\Delta}^2(Y_{\alpha'}^{N_i} \times \mathbf{R}^d)}^2$$
$$+ \sum_{R_{\alpha''}^{N_i} \subset Y_k^{N_j}} \|P_{\xi(R_{\alpha''}^{N_i}), N_i} u\|_{U_{\Delta}^2(R_{\alpha''}^{N_i} \times \mathbf{R}^d)}^2] + \sum_{N_j < N_i} \|P_{\xi(Y_k^{N_j}), N_i} u\|_{U_{\Delta}^2(Y_k^{N_j} \times \mathbf{R}^d)}^2 \lesssim \|u\|_{X(Y_k^{N_j})}^2 + \epsilon^2. \tag{4.6}$$

Finally, for $1 \leq N_i < N_j$, suppose (p, q) is a d-admissible pair.

$$\|P_{\xi(t), N_i} u\|_{L_t^p L_x^q(G_k^{N_j} \times \mathbf{R}^d)} \lesssim (\frac{N_j}{N_i})^{1/p} (\delta^{1/2p} + \epsilon + \|u\|_{\tilde{X}_{N_j}}), \tag{4.7}$$

and

$$\|P_{\xi(t), N_i} u\|_{L_t^p L_x^q(Y_k^{N_j} \times \mathbf{R}^d)} \lesssim (\frac{N_j}{N_i})^{1/p} (\delta^{1/2p} + \epsilon + \|u\|_{\tilde{X}_{N_j}}). \tag{4.8}$$

Proof: See [17].

Also, recall that from [17]

Theorem 4.2 *If $u(t)$ is a minimal mass blowup solution to*

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0, x) &= u_0 \in L^2(\mathbf{R}), \end{aligned} \tag{4.9}$$

$\mu = \pm 1$. There exists a constant C such that for $\epsilon > 0$, $\delta(\epsilon) > 0$ sufficiently small, for any dyadic integer M, if there exist small intervals J_l with

$$[0, T] = \cup_{l=1}^M J_l,$$

$$\sum_{J_l \subset [0, T]} N(J_l) = \frac{\delta M}{2},$$

$$\|u\|_{L_{t,x}^6(J_l \times \mathbf{R})} = \epsilon_0,$$

then

$$\|u\|_{\tilde{X}_M([0, T] \times \mathbf{R})} \leq C\epsilon. \tag{4.10}$$

It was showed in [17] that to prove theorem 4.2 it suffices to prove two intermediate lemmas. Lemmas 4.3 and 4.4 will be proved in this section and the next.

Lemma 4.3 *If $u(t)$ satisfies the conditions in theorem 4.2, then*

$$\|u\|^2_{X(G_k^{N_j})} \lesssim \epsilon^2 \tag{4.11}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{1\leq N_i\leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i}\cap G_k^{N_j}\neq\emptyset} \sum_{\frac{N_i}{32}\leq N_1\leq 32N_i} \|P_{\xi(G_\alpha^{N_i}),N_1} u\|^2_{U^2_\Delta(G_\alpha^{N_i}\times\mathbf{R})} \tag{4.12}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{N_j<N_i} \sum_{\frac{N_i}{32}\leq N_1\leq 32N_i} \|P_{\xi(G_k^{N_j}),N_1} u\|^2_{U^2_\Delta(G_k^{N_j}\times\mathbf{R})} \tag{4.13}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sup_{1\leq N_i<N_j} \sup_{Y_\alpha^{N_i}\cap G_k^{N_j}\neq\emptyset} \sum_{\frac{N_i}{32}\leq N_1\leq 32N_i} \|P_{\xi(Y_\alpha^{N_i}),N_1} u\|^2_{U^2_\Delta(Y_\alpha^{N_i}\times\mathbf{R})} \tag{4.14}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{1\leq N_i\leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i}\cap G_k^{N_j}\neq\emptyset} (\sum_{32N_i<N_1} (\frac{N_i}{N_1})^{1/4} \|P_{\xi(G_\alpha^{N_i}),N_1} u\|_{U^2_\Delta(G_\alpha^{N_i}\times\mathbf{R})})^2 \tag{4.15}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{N_j<N_i} (\sum_{32N_i<N_1} (\frac{N_i}{N_1})^{1/4} \|P_{\xi(G_k^{N_j}),N_1} u\|_{U^2_\Delta(G_k^{N_j}\times\mathbf{R})})^2 \tag{4.16}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sup_{1\leq N_i<N_j} \sup_{Y_\alpha^{N_i}\cap G_k^{N_j}\neq\emptyset} (\sum_{32N_i<N_1} (\frac{N_i}{N_1})^{1/4} \|P_{\xi(Y_\alpha^{N_i}),N_1} u\|_{U^2_\Delta(Y_\alpha^{N_i}\times\mathbf{R})})^2. \tag{4.17}$$

Similarly, we prove

Lemma 4.4 *If $u(t)$ satisfies the conditions in theorem 4.2, then*

$$\|u\|^2_{X(Y_k^{N_j})} \lesssim \epsilon^2 \tag{4.18}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{1\leq N_i\leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i}\cap Y_k^{N_j}\neq\emptyset} \sum_{\frac{N_i}{32}\leq N_1\leq 32N_i} \|P_{\xi(G_\alpha^{N_i}),N_1} u\|^2_{U^2_\Delta(G_\alpha^{N_i}\times\mathbf{R})} \tag{4.19}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{N_j<N_i} \sum_{\frac{N_i}{32}\leq N_1\leq 32N_i} \|P_{\xi(Y_k^{N_j}),N_1} u\|^2_{U^2_\Delta(Y_k^{N_j}\times\mathbf{R})} \tag{4.20}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sup_{1\leq N_i < N_j} \sup_{Y_\alpha^{N_i}\cap Y_k^{N_j}\neq\emptyset} \sum_{\frac{N_i}{32}\leq N_1\leq 32N_i} \|P_{\xi(Y_\alpha^{N_i}),N_1} u\|^2_{U^2_\Delta(Y_\alpha^{N_i}\times\mathbf{R})} \tag{4.21}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{1\leq N_i\leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i}\cap Y_k^{N_j}\neq\emptyset} (\sum_{32N_i<N_1} (\frac{N_i}{N_1})^{1/4} \|P_{\xi(G_\alpha^{N_i}),N_1} u\|_{U^2_\Delta(G_\alpha^{N_i}\times\mathbf{R})})^2 \tag{4.22}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sum_{N_j<N_i} (\sum_{32N_i<N_1} (\frac{N_i}{N_1})^{1/4} \|P_{\xi(Y_k^{N_j}),N_1} u\|_{U^2_\Delta(Y_k^{N_j}\times\mathbf{R})})^2 \tag{4.23}$$

$$+ (\epsilon + \|u\|_{\tilde{X}_{N_j}([0,T]\times\mathbf{R})})^2 \sup_{1\leq N_i\leq N_j} \sup_{Y_\alpha^{N_i}\cap Y_k^{N_j}\neq\emptyset} (\sum_{32N_i<N_1} (\frac{N_i}{N_1})^{1/4} \|P_{\xi(Y_\alpha^{N_i}),N_1} u\|_{U^2_\Delta(Y_\alpha^{N_i}\times\mathbf{R})})^2. \tag{4.24}$$

Start of the proof of lemma 4.3 and lemma 4.4: Take a yellow interval $Y_{\alpha'}^{N_i}$. For any $a_{\alpha',N_i} \in Y_{\alpha'}^{N_i}$, the solution to (1.1) on $Y_{\alpha'}^{N_i}$ is equal to

$$e^{i(t-a_{\alpha',N_i})\Delta} u(a_{\alpha',N_i}) - i\int_{a_{\alpha'}^{N_i}}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau. \tag{4.25}$$

We will postpone the treatment of the Duhamel term,

$$\int_{a_{\alpha'}^{N_i}}^t e^{i(t-\tau)\Delta} |u(\tau)|^4 u(\tau) d\tau,$$

until the next section. Since $N(t) \leq \frac{\delta N_i}{2}$ on $Y_\alpha^{N_i}$,

$$\|P_{\xi(Y_\alpha^{N_i}),\frac{N_i}{4}\leq\cdot\leq 4N_i} u(a_{\alpha,N_i})\|_{L^2_x(\mathbf{R})} \leq \epsilon.$$

Next take $G_L^{N_i}$ and $G_R^{N_i}$. For $a_{L,N_i} \in G_L^{N_i}$ and $a_{R,N_i} \in G_R^{N_i}$,

$$\sum_{1\leq N_i<N_j} (\frac{N_i}{N_j})(\|P_{\xi(G_L^{N_i},\frac{N_i}{4}\leq\cdot\leq 4N_i} u(a_{L,N_i})\|^2_{L^2_x(\mathbf{R})} + \|P_{G_R^{N_i},\frac{N_i}{4}\leq\cdot\leq 4N_i} u(a_{R,N_i})\|^2_{L^2_x(\mathbf{R})}) \lesssim \epsilon^2.$$

Finally, if $G_\alpha^{N_i} \subset G_k^{N_j}$ is a length green interval, we can choose a_{α,N_i} such that

$$\|P_{\xi(G_\alpha^{N_i}),\frac{N_i}{4}\leq\cdot\leq 4N_i} u(a_{\alpha,N_i})\|^2_{L^2_x(\mathbf{R})} \leq \frac{1}{N_i} \sum_{J_l\subset G_\alpha^{N_i}} \|P_{\xi(a_l),\frac{N_i}{8}\leq\cdot\leq 8N_i} u(a_l)\|^2_{L^2_x(\mathbf{R})}, \tag{4.26}$$

where $J_l = [a_l, b_l]$. If G_{α,N_i} is a weight green interval then we can choose $a_{\alpha,N_i} \in G_{\alpha,N_i}$ such that

$$\|P_{\xi(G_\alpha^{N_i}, \frac{N_i}{4} \leq \cdot \leq 4N_i} u(a_{\alpha,N_i})\|_{L_x^2(\mathbf{R}^2)}^2 \leq \frac{2}{\delta N_i} \sum_{J_l \subset G_\alpha^{N_i}} N(J_l) \|P_{\xi(a_l), \frac{N_i}{8} \leq \cdot \leq 8N_i} u(a_l)\|_{L_x^2(\mathbf{R})}^2. \tag{4.27}$$

Therefore,

$$\sum_{1 \leq N_i \leq N_j} (\frac{N_i}{N_j}) \sum_{G_\alpha^{N_i} \cap G_k^{N_j} \neq \emptyset} \|P_{\xi(G_\alpha^{N_i}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u(a_{\alpha,N_i})\|_{L_x^2(\mathbf{R})}^2$$

$$\leq \frac{1}{N_j} \sum_{J_l \subset G_k^{N_j}} \sum_{1 \leq N_i \leq N_j, \frac{2N(J_l)}{\delta} \leq N_i} (1 + \frac{N(J_l)}{\delta}) \|P_{\xi(a_l), \frac{N_i}{8} \leq \cdot \leq 8N_i} u(a_l)\|_{L_x^2(\mathbf{R})}^2$$

$$\lesssim \frac{1}{N_j} \sum_{J_l \subset G_k^{N_j}} (1 + \frac{N(J_l)}{\delta}) \epsilon^2 \lesssim \epsilon^2.$$

Also,

$$\sup_{1 \leq N_i \leq N_j} \sup_{Y_{\alpha'}^{N_i} \cap G_k^{N_j} \neq \emptyset} \|P_{\xi(Y_{\alpha'}^{N_i}), \frac{N_i}{4} \leq \cdot \leq 4N_i} u(a_{\alpha',N_i})\|_{L_x^2(\mathbf{R})}^2 \lesssim \epsilon^2. \tag{4.28}$$

5 Duhamel Terms

Now we turn to the Duhamel term

$$\int_{a_\alpha^{N_i}}^t e^{i(t-\tau)\Delta} P_{\xi(G_\alpha^{N_i}), \frac{N_1}{4} \leq \cdot \leq 4N_1} (|u(\tau)|^4 u(\tau)) d\tau, \tag{5.1}$$

for $N_1 \geq N_i$. We will need the case when $N_1 >> N_i$ for §6. The arguments for $Y_{\alpha'}^{N_i}$ will be virtually identical to the arguments for $G_\alpha^{N_i}$.

$$\begin{aligned} P_{\xi(G_\alpha^{N_i}), \frac{N_1}{4} \leq \cdot \leq 4N_1} (|u(\tau)|^4 u(\tau)) = \\ O((P_{\xi(G_\alpha^{N_i}), \geq \frac{N_1}{32}} u)(P_{\xi(G_\alpha^{N_i}), \geq 2^{-10} N_i} u) u^3) + 5(P_{\xi(G_\alpha^{N_i}), \frac{N_1}{32} \leq \cdot \leq 32N_1} u)(P_{\xi(G_\alpha^{N_i}), \leq 2^{-10} N_i} u)^4. \end{aligned} \tag{5.2}$$

We start with $O((P_{\xi(G_\alpha^{N_i}), \geq \frac{N_1}{32}} u)(P_{\xi(G_\alpha^{N_i}), \geq 2^{-10} N_i} u) u^3)$.

Theorem 5.1 *Suppose $N_1, N_2 \geq \frac{N_i}{32}$, and $G_{\alpha}^{N_i}$ is a green interval.*

$$\begin{aligned} & \| |P_{\xi(G_{\alpha}^{N_i}), N_1} u| |P_{\xi(G_{\alpha}^{N_i}), N_2} u| |P_{\xi(G_{\alpha}^{N_i}), \leq 2^{-10} N_i} u|^4 \|_{L_{t,x}^1(G_{\alpha}^{N_i} \times \mathbf{R})} \\ & \lesssim (\epsilon^2 + \|u\|_{\tilde{X}_{N_i}}^2) \|P_{\xi(G_{\alpha}^{N_i}), N_1} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})} \|P_{\xi(G_{\alpha}^{N_i}), N_2} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})}. \end{aligned} \tag{5.3}$$

Proof: Make a Littlewood - Paley decomposition. Without loss of generality suppose $\xi(G_{\alpha}^{N_i}) = 0$.

$$(5.3) \leq \sum_{N_4 \leq 2^{-10} N_i} \| |P_{N_1} u| |P_{N_2} u| |P_{\xi(t), N_4 \leq \cdot \leq 2^{-10} N_i} u| |P_{\xi(t), N_4} u| |P_{\xi(t), \leq N_4} u|^2 \|_{L_{t,x}^1(G_{\alpha}^{N_i} \times \mathbf{R})}. \tag{5.4}$$

Making a bilinear estimate and using $\|u\|_{U_{\Delta}^2(J_l \times \mathbf{R})} \lesssim_{m_0} 1$,

$$\| |P_{N_1} u| |P_{N_2} u| |P_{\xi(t), N_4 \leq \cdot \leq 2^{-10} N_i} u| |P_{\xi(t), \leq 1} u|^3 \|_{L_{t,x}^1(J_l \times \mathbf{R})}$$

$$\lesssim \| |P_{N_1} u| |P_{\leq 2^{-10} N_i} u| \|_{L_{t,x}^2(J_l \times \mathbf{R})} \| |P_{N_2} u| |P_{\leq 2^{-10} N_i} u| \|_{L_{t,x}^2(J_l \times \mathbf{R})} \|P_{\xi(t), \leq 1} u\|_{L_{t,x}^{\infty}(J_l \times \mathbf{R})}^2$$

$$\begin{aligned} & \lesssim \frac{1}{N_1^{1/2} N_2^{1/2}} (\|P_{\xi(t), \leq \frac{N(t)}{\delta^{1/2}}} u\|_{L_{t,x}^{\infty}(J_l \times \mathbf{R})}^2 + \|P_{\xi(t), \frac{N(t)}{\delta^{1/2}} \leq \cdot \leq 1} u\|_{L_{t,x}^{\infty}(J_l \times \mathbf{R})}^2) \\ & \quad \times \|P_{\xi(G_{\alpha}^{N_i}), N_1} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})} \|P_{\xi(G_{\alpha}^{N_i}), N_2} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})}. \end{aligned}$$

$$\lesssim \frac{1}{N_1^{1/2} N_2^{1/2}} (\frac{N(J_l)}{\delta^{1/2}} + \frac{N(J_l)}{\delta} \epsilon^2) \|P_{\xi(G_{\alpha}^{N_i}), N_1} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})} \|P_{\xi(G_{\alpha}^{N_i}), N_2} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})}.$$

Summing over the subintervals J_l,

$$\begin{aligned} & \| |P_{N_1} u| |P_{N_2} u| |P_{\xi(t), \leq 2^{-10} N_i} u| |P_{\xi(t), \leq 1} u|^3 \|_{L_{t,x}^1(G_{\alpha}^{N_i} \times \mathbf{R})} \\ & \lesssim (\epsilon^2 + \|u\|_{\tilde{X}_{N_i}}^2) \|P_{N_1} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})} \|P_{N_2} u\|_{U_{\Delta}^2(G_{\alpha}^{N_i} \times \mathbf{R})}. \end{aligned} \tag{5.5}$$

Now we consider the case when $N_4 \geq 1$. First take the intervals $R_{\beta''}^{N_4} \subset G_{\alpha}^{N_i}$.

$$\sum_{R_{\beta''}^{N_4} \subset G_{\alpha}^{N_i}} \sum_{1 \leq N_4 \leq 2^{-10} N_i} \| |P_{N_1} u| |P_{N_2} u| |P_{\xi(t), N_4 \leq \cdot \leq 2^{-10} N_i} u| |P_{\xi(t), N_4} u| |P_{\xi(t), \leq N_4} u|^2 \|_{L_{t,x}^1(R_{\beta''}^{N_4} \times \mathbf{R})}$$

$$\lesssim \sum_{R_{\beta''}^{N_4} \subset G_{\alpha}^{N_i}} \sum_{1 \leq N_4 \leq 2^{-10} N_i} \| |P_{N_1} u| |P_{\leq 2^{-10} N_i} u| \|_{L_{t,x}^2(R_{\beta''}^{N_i} \times \mathbf{R})}$$

$$\times \| |P_{N_2} u| |P_{\leq 2^{-10} N_i} u| \|_{L^2_{t,x}(R^{N_i}_{\beta''} \times \mathbf{R})} \| P_{\xi(t), \leq N_4} u \|^2_{L^\infty_{t,x}(R^{N_4}_{\beta''} \times \mathbf{R})}$$

$$\lesssim \frac{1}{N_1^{1/2} N_2^{1/2}} \| P_{N_1} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})} \| P_{N_2} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})} \sum_{1 \leq N_4 \leq 2^{-10} N_i} \sum_{R^{N_4}_{\beta''} \subset G^{N_i}_\alpha} \| P_{\xi(t), \leq N_4} u \|^2_{L^\infty_{t,x}(R^{N_4}_{\beta''} \times \mathbf{R})}$$

$$\lesssim \frac{1}{N_1^{1/2} N_2^{1/2}} \| P_{N_1} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})} \| P_{N_2} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})} \sum_{J_l \subset G^{N_i}_\alpha} (\frac{N(J_l)}{\delta^{1/2}} | \ln(\delta) | + \| P_{\xi(t), \frac{N(t)}{\delta^{1/2}} \leq \cdot \leq \frac{N(t)}{\delta}} u \|^2_{L^\infty_t L^2_x(J_l \times \mathbf{R})})$$

$$\lesssim (\epsilon^2 + \delta^{1/3}) (\frac{N_i}{N_1^{1/2} N_2^{1/2}}) \| P_{N_1} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})} \| P_{N_2} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})}.$$

Next take the intervals $G^{N_4}_\beta$. Let $\tilde{G}^{N_4}_\beta = G^{N_4}_\beta \cap G^{N_i}_\alpha$.

$$\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \sum_{G^{N_4}_\beta \cap G^{N_i}_\alpha \neq \emptyset} \| P_{N_1} u| |P_{N_2} u| |P_{\xi(t), N_3} u| |P_{\xi(t), N_4} u| |P_{\xi(t), \leq N_4} u|^2 \|_{L^1_{t,x}(\tilde{G}^{N_4}_\beta \times \mathbf{R})}$$

$$\lesssim \sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} (\sum_{G^{N_4}_\beta \cap G^{N_i}_\alpha \neq \emptyset} \| |P_{N_1} u| |P_{\xi(G^{N_4}_\beta), \frac{N_4}{4} \leq \cdot \leq 4 N_4} u| \|^2_{L^2_{t,x}(G^{N_4}_\beta \times \mathbf{R})})^{1/2}$$

$$\times \| |P_{N_2} u| |P_{\xi(t), N_3} u| |P_{\xi(t), \leq N_4} u|^2 \|_{L^2_{t,x}(G^{N_i}_\alpha \setminus (\cup R^{N_4}_{\beta''}) \times \mathbf{R})}.$$

$$(\sum_{G^{N_4}_\beta \cap G^{N_i}_\alpha} \| |P_{N_1} u| |P_{\xi(G^{N_4}_\beta), \frac{N_4}{4} \leq \cdot \leq 4 N_4} u| \|^2_{L^2_{t,x}(G^{N_4}_\beta \times \mathbf{R})})^{1/2}$$

$$\lesssim \frac{1}{N_1^{1/2}} \| P_{N_1} u \|_{U^2_\Delta(G^{N_i}_\alpha \times \mathbf{R})} (\sum_{G^{N_4}_\beta \cap G^{N_i}_\alpha \neq \emptyset} \| P_{\xi(G^{N_4}_\beta), \frac{N_4}{4} \leq \cdot \leq 4 N_4} u \|^2_{U^2_\Delta(G^{N_4}_\beta \times \mathbf{R})})^{1/2}.$$

Making bilinear estimates,

$$\| |P_{N_2} u| |P_{\xi(t), N_3} u| |P_{\xi(t), \leq N_4} \|_{L^2_{t,x}(G^{N_i}_\alpha \setminus (\cup R^{N_4}_{\beta''}) \times \mathbf{R})}$$

$$\lesssim N_4 (\sum_{G^{N_3}_\gamma \cap G^{N_i}_\alpha \neq \emptyset} \| |P_{N_2} u| |P_{\xi(G^{N_3}_\gamma), \frac{N_3}{4} \leq \cdot \leq 4 N_3} u| \|^2_{L^2_{t,x}(G^{N_3}_\gamma \times \mathbf{R})})^{1/2}$$

$$+ N_4 (\sum_{Y^{N_3}_{\gamma'} \cap G^{N_i}_\alpha \neq \emptyset} \| |P_{N_2} u| |P_{\xi(Y^{N_3}_{\gamma'}), \frac{N_3}{4} \leq \cdot \leq 4 N_3} u| \|^2_{L^2_{t,x}(Y^{N_3}_{\gamma'} \times \mathbf{R})})^{1/2},$$

by Sobolev embedding this quantity is

$$\lesssim \frac{N_4}{N_2^{1/2}}\|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}(\sum_{G_\gamma^{N_3}}\|P_{\xi(G_\gamma^{N_3}),\frac{N_3}{4}\leq\cdot\leq 4N_3}u\|^2_{U_\Delta^2(G_\gamma^{N_3}\times\mathbf{R})})^{1/2}$$

$$+\frac{N_4}{N_2^{1/2}}\|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}(\sum_{Y_{\gamma'}^{N_3}}\|P_{\xi(Y_{\gamma'}^{N_3}),\frac{N_3}{4}\leq\cdot\leq 4N_3}u\|^2_{U_\Delta^2(Y_{\gamma'}^{N_3}\times\mathbf{R})})^{1/2}.$$

Again by Cauchy - Schwartz,

$$\sum_{1\leq N_4\leq N_3\leq 2^{-10}N_i}(\frac{N_4}{N_3})^{1/2}((\frac{N_4}{N_i})\sum_{G_\beta^{N_4}\cap G_\alpha^{N_i}\neq\emptyset}\|P_{\xi(G_\beta^{N_4}),\frac{N_4}{4}\leq\cdot\leq 4N_4}u\|^2_{U_\Delta^2(G_\beta^{N_4}\times\mathbf{R})})^{1/2}$$

$$\times((\frac{N_3}{N_i})\sum_{G_\gamma^{N_3}\cap G_\alpha^{N_i}\neq\emptyset}\|P_{\xi(G_\gamma^{N_3}),\frac{N_3}{4}\leq\cdot\leq 4N_3}u\|^2_{U_\Delta^2(G_\beta^{N_3}\times\mathbf{R})})^{1/2}\lesssim\|u\|^2_{\tilde{X}_{N_i}}.$$

Next,

$$\sum_{1\leq N_3\leq 2^{-10}N_i}(\frac{N_3}{N_i})\sharp\{Y_{\beta'}^{N_3}\cap G_\alpha^{N_i}\neq\emptyset\}\leq\sum_{1\leq N_3\leq 2^{-10}N_i}(\frac{N_3}{N_i})(\sharp\{R_{\gamma''}^{N_3}\subset G_\alpha^{N_i}\}+1)$$

$$\lesssim 1+\sum_{J_l\subset G_\alpha^{N_i}}\sum_{1\leq N_3\leq\frac{N(J_l)}{\delta}}\frac{N_3}{N_i}\lesssim 1.$$

Because

$$\|P_{\xi(Y_{\gamma'}^{N_3}),\frac{N_3}{4}\leq\cdot\leq 4N_3}u\|_{U_\Delta^2(Y_{\gamma'}^{N_3}\times\mathbf{R})}\lesssim\|u\|_{\tilde{X}_{N_i}},$$

$$\sum_{1\leq N_3\leq 2^{-10}N_i}(\frac{N_3}{N_i})\sum_{Y_{\gamma'}^{N_3}\cap G_\alpha^{N_i}\neq\emptyset}\|P_{\xi(Y_{\gamma'}^{N_3}),\frac{N_3}{4}\leq\cdot\leq 4N_3}u\|^2_{U_\Delta^2(Y_{\gamma'}^{N_3}\times\mathbf{R})}\lesssim\|u\|^2_{\tilde{X}_{N_i}}.$$

Therefore, by Cauchy-Schwartz,

$$\sum_{1\leq N_4\leq N_3\leq 2^{-10}N_i}\sum_{G_\beta^{N_4}\cap G_\alpha^{N_i}\neq\emptyset}\||P_{N_1}u||P_{N_2}u||P_{\xi(t),N_3}u||P_{\xi(t),N_4}u||P_{\xi(t),\leq N_4}u|^2\|_{L^1_{t,x}(G_\beta^{N_4}\times\mathbf{R})}$$
$$\lesssim(\frac{N_i}{N_1^{1/2}N_2^{1/2}})\|u\|^2_{\tilde{X}_{N_i}}\|P_{N_1}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}\|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}. \tag{5.6}$$

Similarly,

$$\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \sum_{Y_{\beta'}^{N_4} \cap G_\alpha^{N_i} \neq \emptyset} \| |P_{N_1} u| |P_{N_2} u| |P_{\xi(t), N_3} u| |P_{\xi(t), N_4} u| |P_{\xi(t), \leq N_4} u|^2 \|_{L^1_{t,x}(Y_{\beta'}^{N_4} \times \mathbf{R})}$$
$$\lesssim (\frac{N_i}{N_1^{1/2} N_2^{1/2}}) \|u\|^2_{\tilde{X}_{N_i}} \|P_{N_1} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})} \|P_{N_2} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})}. \tag{5.7}$$

This completes the proof of the theorem. We could make exactly the same arguments for the yellow interval $Y_\alpha^{N_i}$. $\square$

Corollary 5.2 *Making virtually identical arguments,*

$$\| |P_{N_1} u| |P_{N_2} u| |P_{\leq 2^{-10} N_i} (P_{\xi(t), \geq C_0 N(t)} u)|^4 \|_{L^1_{t,x}(G_\alpha^{N_i} \times \mathbf{R})}$$
$$\lesssim (\frac{N_i}{N_1^{1/2} N_2^{1/2}})(\epsilon^2 + \|u\|^2_{\tilde{X}_{N_i}}) \|P_{N_1} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})} \|P_{N_2} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})} \|u_{\xi(t), \geq C_0 N(t)}\|^2_{L_t^\infty L_x^2(G_\alpha^{N_i} \times \mathbf{R})}. \tag{5.8}$$

$$\| |P_{N_1} u| |P_{N_2} u| |P_{\xi(t), \leq C_0 N(t)} u|^4 \|_{L^1_{t,x}(G_\alpha^{N_i} \times \mathbf{R})}$$
$$\lesssim C_0 (\frac{N_i}{N_1^{1/2} N_2^{1/2}})(\epsilon^2 + \|u\|^2_{\tilde{X}_{N_i}})(\sup_{J_l \subset G_\alpha^{N_i}} \|P_{N_1} u\|_{U^2_\Delta(J_l \times \mathbf{R})})(\sup_{J_l \subset G_\alpha^{N_i}} \|P_{N_2} u\|_{U^2_\Delta(J_l \times \mathbf{R})}). \tag{5.9}$$

Theorem 5.3 *For $N_1 \geq N_i$,*

$$\|P_{N_1}((P_{\geq \frac{N_1}{32}} u)(P_{\geq 2^{-10} N_i} u) u^3)\|_{DU^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})} \lesssim$$
$$(\epsilon^2 + \|u\|^2_{\tilde{X}_{N_i}}) \sum_{N_2 \geq \frac{N_i}{32}} (\frac{N_i}{N_2})^{1/4} \|P_{\xi(G_\alpha^{N_i}), N_2} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})}. \tag{5.10}$$

Proof: Take v supported on $|\xi| \sim N_1$, $\|v\|_{V^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})} = 1$.

$$\int_{G_\alpha^{N_i}} \langle v, (P_{> \frac{N_1}{32}} u)(P_{> 2^{-10} N_i} u) u^3 \rangle dt \leq \| |v| |P_{> \frac{N_1}{32}} u| |P_{> 2^{-10} N_i} u| |u|^3 \|_{L^1_{t,x}(G_\alpha^{N_i} \times \mathbf{R})}$$

$$\lesssim \| |v| |P_{> \frac{N_1}{32}} u| |P_{> 2^{-10} N_i} u| |P_{\geq 2^{-10} N_i} u|^3 \|_{L^1_{t,x}(G_\alpha^{N_i} \times \mathbf{R})} + \| |v| |P_{> \frac{N_1}{32}} u| |P_{> 2^{-10} N_i} u| |P_{\leq 2^{-10} N_i} u|^3 \|_{L^1_{t,x}(G_\alpha^{N_i} \times \mathbf{R})}.$$

$$\| |v| |P_{> \frac{N_1}{32}} u| |P_{> 2^{-10} N_i} u| |P_{\leq 2^{-10} N_i} u|^3 \|_{L^1_{t,x}(G_\alpha^{N_i} \times \mathbf{R})}$$

$$\lesssim \|v\|_{L_t^4 L_x^\infty(G_\alpha^{N_i}\times\mathbf{R})} \|P_{>2^{-10}N_i}u\|_{L_t^4 L_x^\infty(G_\alpha^{N_i}\times\mathbf{R})} \|u\|_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})} \||P_{>\frac{N_1}{32}}u||P_{\leq 2^{-10}N_i}u|^2\|_{L_{t,x}^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

By theorem 5.1,

$$\||P_{>\frac{N_1}{32}}u||P_{\leq 2^{-10}N_i}u|^2\|_{L_{t,x}^2(G_\alpha^{N_i}\times\mathbf{R})} \lesssim (\epsilon + \|u\|_{\tilde{X}_{N_i}}) \sum_{N_2 \geq \frac{N_1}{32}} (\frac{N_i}{N_2})^{1/2} \|P_{\xi(G_\alpha^{N_i}),N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

Therefore,

$$\||v||P_{>\frac{N_1}{32}}u||P_{>2^{-10}N_i}u||P_{\leq 2^{-10}N_i}u|^3\|_{L_{t,x}^1(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim (\epsilon^2 + \|u\|_{\tilde{X}_{N_i}}^2) \sum_{N_2 \geq \frac{N_1}{32}} (\frac{N_i}{N_2})^{1/2} \|P_{\xi(G_\alpha^{N_i}),N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

Next, because $V_\Delta^2 \subset U_\Delta^3$, by (2.23)

$$\|(v)(P_{N_2}u)\|_{L_{t,x}^3(G_\alpha^{N_i}\times\mathbf{R})} \lesssim (\frac{N_1}{N_2})^{1/4} \|v\|_{V_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})} \|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

Therefore,

$$\||v||P_{N_2}u||P_{>2^{-10}N_i}u|^4\|_{L_{t,x}^1(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \|P_{>2^{-10}N_i}u\|_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})} \sum_{N_2 \geq \frac{N_1}{32}} (\frac{N_i}{N_2})^{1/4} \|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})} \|P_{>2^{-10}N_i}u\|_{L_t^{9/2}L_x^{18}(G_\alpha^{N_i}\times\mathbf{R})}^3.$$

$$\lesssim (\epsilon^2 + \|u\|_{\tilde{X}_{N_i}}^2) \sum_{N_2 \geq \frac{N_1}{32}} (\frac{N_i}{N_2})^{1/4} \|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

The proof of theorem 5.3 is complete. $\Box$

Theorem 5.4 *Suppose* $\|u\|_{\tilde{X}_{N_i}} \lesssim 1$. *Then*

$$\begin{aligned} &\||P_{\geq\frac{N_1}{32}}u||P_{\geq 2^{-10}N_i}u||u|^3\|_{DU_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})} \lesssim (\frac{N_i}{N_1})^{1/2} \|P_{\geq\frac{N_1}{32}}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})} \\ &\|P_{\geq\frac{N_1}{32}}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})} (\sum_{2^{-10}N_i \leq N_2 \leq 2^{-10}N_1} (\frac{N_2}{N_1})^{1/4} \|P_{\geq N_2}u\|_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})}^{1/2}). \end{aligned} \tag{5.11}$$

Proof: Take $\|v\|_{V_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}=1$.

$$\||v||P_{\geq\frac{N_1}{32}}u||P_{\geq 2^{-10}N_i}u||P_{\leq 2^{-10}N_i}u|^3\|_{L^1_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \||P_{\geq\frac{N_1}{32}}u||P_{\leq 2^{-10}N_i}u|^2\|_{L^2_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}\|v\|_{L_t^4L_x^\infty(G_\alpha^{N_i}\times\mathbf{R})}\|P_{\geq 2^{-10}N_i}u\|_{L_t^4L_x^\infty(G_\alpha^{N_i}\times\mathbf{R})}\|u\|_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})},$$

which by theorem 5.1, conservation of mass,

$$\lesssim (\frac{N_i}{N_1})^{1/2}\|P_{\geq\frac{N_1}{32}}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

Next,

$$\||v||P_{\geq\frac{N_1}{32}}u||P_{\geq 2^{-10}N_1}u||P_{\geq 2^{-10}N_i}u|^3\|_{L^1_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \|P_{\geq\frac{N_1}{32}}u\|_{L_t^4L_x^\infty(G_\alpha^{N_i}\times\mathbf{R})}\|v\|_{L_t^4L_x^\infty(G_\alpha^{N_i}\times\mathbf{R})}\|P_{\geq 2^{-10}N_1}u\|_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})}\|P_{\geq 2^{-10}N_i}u\|^3_{L^6_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \|P_{\geq\frac{N_1}{32}}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}\|P_{\geq 2^{-10}N_1}u\|_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

Finally, for $2^{-10}N_i\leq N_2\leq 2^{-10}N_1$,

$$\|(P_{\geq\frac{N_1}{32}}u)(P_{N_2}u)\|_{L^3_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \|P_{\geq\frac{N_1}{32}}u\|^{1/2}_{L^6_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}\|(P_{\geq\frac{N_1}{32}}u)(P_{N_2}u)\|^{1/2}_{L^2_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}\|P_{N_2}u\|^{1/2}_{L^\infty_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim (\frac{N_2}{N_1})^{1/4}\|P_{\geq\frac{N_1}{32}}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}\|P_{N_2}u\|^{1/2}_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

Therefore,

$$\||v||P_{\geq\frac{N_1}{32}}u||P_{2^{-10}N_i\leq\cdot\leq 2^{-10}N_1}u||P_{\geq 2^{-10}N_i}u|^3\|_{L^1_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \|v\|_{L^6_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}\|(P_{\geq\frac{N_1}{32}}u)(P_{2^{-10}N_i\leq\cdot\leq 2^{-10}N_1}u)\|_{L^3_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}\|P_{\geq 2^{-10}N_i}u\|^3_{L^6_{t,x}(G_\alpha^{N_i}\times\mathbf{R})}$$

$$\lesssim \sum_{2^{-10}N_i\leq N_2\leq 2^{-10}N_1}(\frac{N_2}{N_1})^{1/4}\|P_{\geq\frac{N_1}{32}}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}\|P_{N_2}u\|^{1/2}_{L_t^\infty L_x^2(G_\alpha^{N_i}\times\mathbf{R})}.$$

This completes the proof of the theorem. □

Theorem 5.5 *For $N_1 \geq N_i$, $G_\alpha^{N_i} = [a_\alpha^{N_i}, b_\alpha^{N_i}]$, $\xi(G_\alpha^{N_i}) = 0$,*

$$\| \int_{a_\alpha^{N_i}}^t e^{i(t-\tau)\Delta} P_{N_1}((P_{N_2}u)(P_{\leq 2^{-10}N_i}u)^4(\tau))d\tau\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})} \lesssim (\frac{N_i}{N_1^{1/2}N_2^{1/2}})\|P_{N_2}u\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}(\epsilon^2+\|u\|^2_{\tilde{X}_{N_i}}). \tag{5.12}$$

Proof: Let $G_\beta^{N_4} = [a_\beta^{N_4}, b_\beta^{N_4}]$, $Y_{\beta'}^{N_4} = [a_{\beta'}^{N_4}, b_{\beta'}^{N_4}]$, $R_{\beta''}^{N_4} = [a_{\beta''}^{N_4}, b_{\beta''}^{N_4}]$. Let

$$u_{nl}^{G_\beta^{N_4},N_3}(t) = \int_{a_\beta^{N_4}}^t e^{i(t-\tau)\Delta}(P_{N_2}u)(P_{\xi(\tau),N_4}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2(\tau)d\tau, \tag{5.13}$$

$$u_{nl}^{Y_{\beta'}^{N_4},N_3}(t) = \int_{a_\beta^{N_4}}^t e^{i(t-\tau)\Delta}(P_{N_2}u)(P_{\xi(\tau),N_4}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2(\tau)d\tau, \tag{5.14}$$

$$u_{nl}^{R_{\beta''}^{N_4}}(t) = \int_{a_\beta^{N_4}}^t e^{i(t-\tau)\Delta}(P_{N_2}u)(P_{\xi(\tau),N_4}u)(P_{\xi(\tau),N_4\leq\cdot\leq 2^{-10}N_i}u)(P_{\xi(\tau),\leq N_4}u)^2(\tau)d\tau. \tag{5.15}$$

Then

$$(5.12) \lesssim \sum_{1\leq N_4\leq N_3\leq 2^{-10}N_i}[\sum_{G_\beta^{N_4}\cap G_\alpha^{N_i}}\|u_{nl}^{G_\beta^{N_4},N_3}(b_\beta^{N_4})\|_{L_x^2(\mathbf{R})} + \sum_{Y_{\beta'}^{N_4}\cap G_\alpha^{N_i}}\|u_{nl}^{Y_{\beta'}^{N_4},N_3}(b_{\beta'}^{N_4})\|_{L_x^2(\mathbf{R})}] \tag{5.16}$$

$$+ \sum_{1\leq N_4\leq 2^{-10}N_i}[\sum_{R_{\beta''}^{N_4}\subset G_\alpha^{N_i}}\|u_{nl}^{R_{\beta''}^{N_4}}(b_{\beta''}^{N_4})\|_{L_x^2(\mathbf{R})} + (\sum_{R_{\beta''}^{N_4}\subset G_\alpha^{N_i}}\|u_{nl}^{R_{\beta''}^{N_4}}(t)\|^2_{U_\Delta^2(R_{\beta''}^{N_4})})^{1/2}] \tag{5.17}$$

$$+ \sum_{1\leq N_4\leq N_3\leq 2^{-10}N_i}[(\sum_{G_\beta^{N_4}\cap G_\alpha^{N_i}}\|u_{nl}^{G_\beta^{N_4},N_3}(b_\beta^{N_4})\|^2_{U_\Delta^2(G_\beta^{N_4}\times\mathbf{R})})^{1/2}+(\sum_{Y_{\beta'}^{N_4}\cap G_\alpha^{N_i}}\|u_{nl}^{Y_{\beta'}^{N_4},N_3}(b_{\beta'}^{N_4})\|^2_{U_\Delta^2(Y_{\beta'}^{N_4}\times\mathbf{R})})^{1/2}] \tag{5.18}$$

$$+ \| \int_{a_\alpha^{N_i}}^t e^{i(t-\tau)\Delta}(P_{N_2}u)(P_{\xi(\tau),\leq 1}u)^3u(\tau)d\tau\|_{U_\Delta^2(G_\alpha^{N_i}\times\mathbf{R})}. \tag{5.19}$$

Take $\|F\|_{L^2(\mathbf{R})}$ supported on $|\xi| \sim N_1$.

$$\|P_{N_1}(\int_{a_\beta^{N_4}}^{b_\beta^{N_4}} e^{i(b_\beta^{N_4}-\tau)\Delta}(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),N_4}u)(P_{\xi(\tau),\leq N_4}u)^2(\tau)d\tau)\|_{L_x^2(\mathbf{R})}$$

$$= \sup_{\|F\|_{L^2(\mathbf{R})}=1} \int_{a_\beta^{N_4}}^{b_\beta^{N_4}} \langle F, e^{i(b_\beta^{N_4}-\tau)\Delta}(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),N_4}u)(P_{\xi(\tau),\leq N_4}u)^2(\tau)d\tau\rangle$$

$$= \int_{a_\beta^{N_4}}^{b_\beta^{N_4}} \langle e^{i(\tau-b_\beta^{N_4})\Delta}F, (P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),N_4}u)(P_{\xi(\tau),\leq N_4}u)^2\rangle d\tau$$

$$\lesssim \|(e^{i(\tau-b_\beta^{N_4})\Delta}F)(P_{\xi(G_\beta^{N_4}),\frac{N_4}{4}\leq\cdot 4N_4}u)\|_{L^2_{t,x}(G_\beta^{N_4}\times\mathbf{R})}\|(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2\|_{L^2_{t,x}(G_\beta^{N_4}\times\mathbf{R})}.$$

Making a bilinear estimate,

$$\|(e^{i(\tau-b_\beta^{N_4})\Delta}F)(P_{\xi(G_\beta^{N_4}),\frac{N_4}{4}\leq\cdot 4N_4}u)\|_{L^2_{t,x}(G_\beta^{N_4}\times\mathbf{R})} \lesssim \frac{1}{N_2^{1/2}}\|P_{\xi(G_\beta^{N_4}),\frac{N_4}{4}\leq\cdot 4N_4}u\|_{U^2_\Delta(G_\beta^{N_4}\times\mathbf{R})}.$$

By Holder's inequality,

$$\sum_{1\leq N_4\leq N_3\leq 2^{-10}N_i}\sum_{G_\beta^{N_4}\cap G_\alpha^{N_i}}\|u_{nl}^{G_\beta^{N_4},N_3}(b_\beta^{N_4})\|_{L^2_x(\mathbf{R})}$$

$$\lesssim \sum_{1\leq N_4\leq N_3\leq 2^{-10}N_i}\frac{1}{N_2^{1/2}}(\sum_{G_\beta^{N_4}\cap G_\alpha^{N_i}\neq\emptyset}\|P_{\xi(G_\beta^{N_4}),\frac{N_4}{4}\leq\cdot 4N_4}u\|^2_{U^2_\Delta(G_\beta^{N_4}\times\mathbf{R})})^{1/2}$$

$$\times\|(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2\|_{L^2_{t,x}((G_\alpha^{N_i}\setminus(\cup R_{\beta''}^{N_4}))\times\mathbf{R})}.$$

$$\|(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2\|_{L^2_{t,x}(G_\alpha^{N_i}\setminus(\cup R_{\beta''}^{N_4})\times\mathbf{R})}$$

$$\lesssim (\sum_{G_\gamma^{N_3}\cap G_\alpha^{N_i}\neq\emptyset}\|(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2\|^2_{L^2_{t,x}(G_\gamma^{N_3}\times\mathbf{R})})^{1/2}$$

$$+(\sum_{Y_{\gamma'}^{N_3}}\|(P_{N_2}u)(P_{\xi(\tau),N_3}u)(P_{\xi(\tau),\leq N_4}u)^2\|^2_{L^2_{t,x}(G_\gamma^{N_3}\times\mathbf{R})})^{1/2}.$$

$$\|(P_{N_2}u)(P_{\xi(\tau),N_3}u)\|_{L^2_{t,x}(G_\beta^{N_3}\times\mathbf{R})} \lesssim \frac{1}{N_2^{1/2}}\|P_{N_2}u\|_{U^2_\Delta(G_\alpha^{N_i}\times\mathbf{R})}\|P_{\xi(G_\beta^{N_3}),\frac{N_3}{4}\leq\cdot\leq 4N_3}u\|_{U^2_\Delta(G_\beta^{N_3}\times\mathbf{R})}.$$

Therefore, as in the proof of theorem 5.1,

$$\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \sum_{G_\beta^{N_4} \cap G_\alpha^{N_i} \neq \emptyset} \| u_{nl}^{G_\beta^{N_4}, N_3}(b_\beta^{N_4}) \|_{L_x^2(\mathbf{R})}$$

$$\lesssim \| P_{N_2} u \|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \Big(\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} (\frac{N_4}{N_3})^{1/2} ((\frac{N_4}{N_2}) \sum_{G_\beta^{N_4} \cap G_\alpha^{N_i} \neq \emptyset} \| P_{\xi(G_\beta^{N_4}), \frac{N_4}{4} \leq \cdot \leq 4 N_4} u \|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}^2)^{1/2}$$

$$\times ((\frac{N_3}{N_2}) \sum_{G_\gamma^{N_3} \cap G_\alpha^{N_i} \neq \emptyset} \| P_{\xi(G_\gamma^{N_3}), \frac{N_3}{4} \leq \cdot \leq 4 N_3} u \|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}^2)^{1/2} \Big)$$

$$+ \| P_{N_2} u \|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \Big(\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} (\frac{N_4}{N_3})^{1/2} ((\frac{N_4}{N_2}) \sum_{G_\beta^{N_4} \cap G_\alpha^{N_i} \neq \emptyset} \| P_{\xi(G_\beta^{N_4}), \frac{N_4}{4} \leq \cdot \leq 4 N_4} u \|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}^2)^{1/2}$$

$$\times ((\frac{N_3}{N_2}) \sum_{Y_{\gamma'}^{N_3} \cap G_\alpha^{N_i} \neq \emptyset} \| P_{\xi(Y_{\gamma'}^{N_3}), \frac{N_3}{4} \leq \cdot \leq 4 N_3} u \|_{U_\Delta^2(G_\gamma^{N_3} \times \mathbf{R})}^2)^{1/2} \Big)$$

$$\lesssim \| P_{N_2} u \|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \| u \|_{\tilde{X}_{N_i}}^2.$$

Similarly,

$$\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \sum_{Y_{\beta'}^{N_4} \cap G_\alpha^{N_i} \neq \emptyset} \| u_{nl}^{Y_{\beta'}^{N_4}, N_3}(b_{\beta'}^{N_4}) \|_{L_x^2(\mathbf{R})} \lesssim \| P_{N_2} u \|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \| u \|_{\tilde{X}_{N_i}}^2.$$

This takes care of (5.16). Next take $R_{\beta''}^{N_4} = [a_{\beta''}^{N_4}, b_{\beta''}^{N_4}]$.

$$\int_{a_{\beta''}^{N_4}}^{b_{\beta''}^{N_4}} \langle e^{i(\tau - b_{\beta''}^{N_4})\Delta} F, (P_{N_2} u)(P_{\xi(\tau), \leq \cdot 2^{-10} N_i} u)(P_{\xi(\tau), N_4} u)(P_{\xi(\tau), \leq N_4} u)^2 \rangle d\tau$$

$$\lesssim \| (e^{i(\tau - b_{\beta''}^{N_4} \Delta} F)(P_{\leq 2^{-10} N_i} u) \|_{L_{t,x}^2(R_{\beta''}^{N_4} \times \mathbf{R})} \| (P_{N_2} u)(P_{\leq 2^{-10} N_i} u) \|_{L_{t,x}^2(R_{\beta''}^{N_4} \times \mathbf{R})} \| P_{\xi(\tau), \leq N_4} u \|_{L_{t,x}^\infty(R_{\beta''}^{N_4} \times \mathbf{R})}^2$$

$$\lesssim \frac{1}{N_2} \| P_{N_2} u \|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \| P_{\xi(\tau), \leq N_4} u \|_{L_{t,x}^\infty(R_{\beta''}^{N_4} \times \mathbf{R})}^2.$$

Therefore,

$$\sum_{1 \leq N_4 \leq 2^{-10} N_i} \sum_{R_{\beta''}^{N_4} \subset G_\alpha^{N_i}} \|u_{nl}^{R_{\beta''}^{N_4}}(b_{\beta''}^{N_4})\|_{L_x^2(\mathbf{R})} \lesssim \frac{1}{N_2} \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \sum_{J_l \subset G_\alpha^{N_i}} \|P_{\xi(t), \leq \frac{N(J_l)}{\delta}} u\|_{L_{t,x}^\infty(J_l \times \mathbf{R})}$$

$$\lesssim \frac{N_i}{N_2} \epsilon^2 \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})}.$$

Now take v supported on $|\xi| \sim N_2$, $\|v\|_{V_\Delta^2(R_{\beta''}^{N_2} \times \mathbf{R})} = 1$.

$$\int_{R_{\beta''}^{N_4}} \langle v, (P_{N_2} u)(P_{\xi(t), N_4 \leq \cdot \leq 2^{-10} N_i} u)(P_{\xi(t), N_4} u)(P_{\xi(t), \leq N_4} u)^2 \rangle dt$$

$$\lesssim \|(P_{N_2} u)(P_{\leq 2^{-10} N_i} u)\|_{L_{t,x}^2(R_{\beta''}^{N_4} \times \mathbf{R})} \|(v)(P_{\leq 2^{-10} N_i} u)\|_{L_{t,x}^{5/2}(R_{\beta''}^{N_4} \times \mathbf{R})} \|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^{20}(R_{\beta''}^{N_4} \times \mathbf{R})}^2$$

$$\lesssim (\frac{N_4}{N_2})^{7/10} \epsilon^{7/5} \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^\infty(R_{\beta''}^{N_4} \times \mathbf{R})}^{7/5}.$$

We interpolated $\|u\|_{L_{t,x}^6(R_{\beta''}^{N_4} \times \mathbf{R})} \lesssim 1$ with $\|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^\infty}$. By Sobolev embedding and conservation of mass,

$$\sum_{1 \leq N_4 \leq 2^{-10} N_i} \frac{1}{N_i^{7/10}} (\sum_{R_{\beta''}^{N_4} \subset G_\alpha^{N_i}} \|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^\infty(R_{\beta''}^{N_4} \times \mathbf{R})}^{14/5})^{1/2}$$

$$\lesssim \sum_{1 \leq N_4 \leq 2^{-10} N_i} (\frac{N_4}{N_i})^{1/5} (\sum_{R_{\beta''}^{N_4} \subset G_\alpha^{N_i}} \frac{1}{N_i} \|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^\infty(R_{\beta''}^{N_4} \times \mathbf{R})})^{1/2}.$$

By Holders inequality,

$$\lesssim (\sum_{1 \leq N_4 \leq 2^{-10} N_i} \sum_{R_{\beta''}^{N_4} \subset G_\alpha^{N_i}} (\frac{N_4}{N_i})^{2/5})^{1/2} (\frac{1}{N_i} \sum_{1 \leq N_4 \leq 2^{-10} N_i} \|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^\infty(R_{\beta''}^{N_4} \times \mathbf{R})})^{1/2} \lesssim (\frac{N_i}{N_2})^{7/10} \epsilon.$$

Therefore,

$$\sum_{1 \leq N_4 \leq 2^{-10} N_i} (\sum_{R_{\beta''}^{N_4} \subset G_\alpha^{N_i}} \|u_{nl}^{R_{\beta''}^{N_4}}(t)\|_{U_\Delta^2(R_{\beta''}^{N_4} \times \mathbf{R})}^2)^{1/2} \lesssim (\frac{N_i}{N_2})^{7/10} \epsilon \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})}.$$

This takes care of (5.17).

Next, for $G_\beta^{N_4} = [a_\beta^{N_4}, b_\beta^{N_4}]$,

$$\int_{a_\beta^{N_4}}^{b_\beta^{N_4}} \langle v, (P_{N_2} u)(P_{\xi(t), N_3} u)(P_{\xi(t), N_4} u)(P_{\xi(t), \leq N_4} u)^2 \rangle dt$$

$$\lesssim \|v(P_{\xi(G_\beta^{N_4}), \frac{N_4}{4} \leq \cdot \leq 4N_4})\|_{L_{t,x}^{5/2}(G_\beta^{N_4} \times \mathbf{R})} \|(P_{N_2} u)(P_{\xi(t), N_3} u)\|_{L_{t,x}^2(G_\beta^{N_4} \times \mathbf{R})} \|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^{20}(G_\beta^{N_4} \times \mathbf{R})}^2.$$

By lemma 4.1, Sobolev embedding,

$$\|P_{\xi(t), \leq N_4} u\|_{L_{t,x}^{20}(G_\beta^{N_4} \times \mathbf{R})}^2 \lesssim N_4^{7/10} (1 + \|u\|_{\tilde{X}_{N_i}})^2.$$

Because $V_\Delta^2 \subset U_\Delta^{5/2}$,

$$\|v(P_{\xi(G_\beta^{N_4}), \frac{N_4}{4} \leq \cdot \leq 4N_4})\|_{L_{t,x}^{5/2}(G_\beta^{N_4} \times \mathbf{R})} \lesssim \frac{1}{N_2^{1/5}} \|P_{\xi(G_\beta^{N_4}), \frac{N_4}{4} \leq \cdot \leq 4N_4}\|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}.$$

Finally, because $G_\beta^{N_4}$ overlaps at most two green intervals at level N_3 and at most two yellow intervals at level N_3,

$$\|(P_{\xi(t), N_3} u)(P_{N_2} u)\|_{L_{t,x}^2(G_\beta^{N_4} \times \mathbf{R})} \lesssim \frac{1}{N_2^{1/2}} \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \|u\|_{\tilde{X}_{N_i}}.$$

Therefore,

$$\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \Big(\sum_{G_\beta^{N_4} \cap G_\alpha^{N_i}} \|u_{nl}^{G_\beta^{N_4}, N_3}(t)\|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}^2 \Big)^{1/2}$$

$$\lesssim \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \|u\|_{\tilde{X}_{N_i}} \sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \frac{N_4^{1/5}}{N_2^{1/5}} \Big(\Big(\frac{N_4}{N_2}\Big) \sum_{G_\beta^{N_4} \cap G_\alpha^{N_i} \neq \emptyset} \|P_{\xi(G_\beta^{N_4}), \frac{N_4}{4} \leq \cdot \leq 4N_4} u\|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}^2 \Big)^{1/2}$$

$$\lesssim \Big(\frac{N_i}{N_2}\Big)^{1/2} \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \|u\|_{\tilde{X}_{N_i}}^2 \sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \frac{N_4^{1/5}}{N_2^{1/5}} \lesssim \Big(\frac{N_i}{N_2}\Big)^{7/10} \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \|u\|_{\tilde{X}_{N_i}}^2.$$

Similarly, using $\sharp\{Y_{\beta'}^{N_4} \cap G_\alpha^{N_i}\} \lesssim \frac{N_i}{N_4}$,

$$\sum_{1 \leq N_4 \leq N_3 \leq 2^{-10} N_i} \Big(\sum_{G_\beta^{N_4} \cap G_\alpha^{N_i}} \|u_{nl}^{Y_{\beta'}^{N_4}, N_3}(t)\|_{U_\Delta^2(G_\beta^{N_4} \times \mathbf{R})}^2 \Big)^{1/2} \lesssim \Big(\frac{N_i}{N_2}\Big)^{7/10} \|P_{N_2} u\|_{U_\Delta^2(G_\alpha^{N_i} \times \mathbf{R})} \|u\|_{\tilde{X}_{N_i}}^2.$$

This takes care of (5.18). Finally, for $\hat{F}$ supported on $|\xi| \sim N_2$, $\|F\|_{L^2(\mathbf{R})} = 1$,

$$\int_{a_l}^{b_l} \langle e^{i(t-b_l)\Delta} F, (P_{N_2} u)(P_{\leq 2^{-10} N_i} u)(P_{\xi(t), \leq 1} u)^3 \rangle dt$$

$$\lesssim \|(e^{i(t-b_l)\Delta} F)(P_{\leq 2^{-10} N_i} u)\|_{L^2_{t,x}(J_l \times \mathbf{R})} \|(P_{N_2} u)(P_{\leq 2^{-10} N_i} u)\|_{L^2_{t,x}(J_l \times \mathbf{R})}$$

$$\times [\|P_{\xi(t), \leq \frac{N(t)}{\delta^{1/2}}} u\|^2_{L^\infty_{t,x}(J_l \times \mathbf{R})} + \|P_{\xi(t), \frac{N(t)}{\delta^{1/2}} \leq \cdot \leq 1} u\|_{L^\infty_{t,x}(J_l \times \mathbf{R})}].$$

$$\lesssim \frac{N(J_l)}{N_2} \epsilon \|P_{N_2} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})}.$$

$$\int_{J_l} \langle v, (P_{N_2} u)(P_{\leq 2^{-10} N_i} u)(P_{\xi(t), \leq 1} u)^3 \rangle dt$$

$$\lesssim \|(P_{N_2} u)(P_{\leq 2^{-10} N_i} u)\|_{L^2_{t,x}(J_l \times \mathbf{R})} \|v(P_{\leq 2^{-10} N_i} u)\|_{L^{5/2}_{t,x}(J_l \times \mathbf{R})}$$

$$\times [\|P_{\xi(t), \leq \frac{N(t)}{\delta^{1/2}}} u\|^2_{L^{20}_{t,x}(J_l \times \mathbf{R})} + \|P_{\xi(t), \frac{N(t)}{\delta^{1/2}} \leq \cdot \leq 1} u\|^2_{L^{20}_{t,x}(J_l \times \mathbf{R})}$$

$$\lesssim \epsilon^{7/5} \frac{N(J_l)}{N_2}^{7/10} \|P_{N_2} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})}.$$

Let

$$u_{nl}^{J_l, \leq 1}(t) = \int_{a_l}^t e^{i(t-\tau)\Delta} (P_{N_2} u)(P_{\leq 2^{-10} N_i} u)(P_{\xi(\tau), 1} u)(\tau) d\tau.$$

$$\| \int_{a_\alpha^{N_i}}^t e^{i(t-\tau)\Delta} (P_{N_2} u)(P_{\leq 2^{-10} N_i} u)(P_{\xi(t), \leq 1} u)^3(\tau) d\tau \|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})}$$

$$\lesssim \sum_{J_l \subset G_\alpha^{N_i}} \|u_{nl}^{J_l, \leq 1}(b_l)\|_{L^2_x(\mathbf{R})} + (\sum_{J_l \subset G_\alpha^{N_i}} \|u_{nl}^{J_l, \leq 1}\|^2_{U^2_\Delta(J_l \times \mathbf{R})})^{1/2}$$

$$\lesssim \epsilon^{7/5} (\frac{N_i}{N_2})^{7/10} \|P_{N_2} u\|_{U^2_\Delta(G_\alpha^{N_i} \times \mathbf{R})}.$$

We have finished the proof of theorem 5.3. □

We combine theorems 5.3 and 5.5 to estimate the Duhamel terms for $G_\alpha^{N_i}$. We apply theorem 5.3 to estimate the first term in (5.2) and theorem 5.5 to estimate the second term in (5.2). The estimates of the Duhamel terms for $Y_{\alpha'}^{N_i}$ follow in identical fashion. Therefore, the proof of lemmas 4.3 and 4.4, and consequently theorem 4.2, is complete.

6 The case when $\int_0^\infty N(t)^3 dt < \infty$

In this section we prove

Theorem 6.1 *There does not exist a one dimensional minimal mass blowup solution to (1.1), $\mu = +1$, with $N(t) \leq 1$,*

$$\int_0^\infty N(t)^3 dt < \infty.$$

To prove this we prove an intermediate theorem.

Theorem 6.2 *Suppose $u(t,x)$ is a minimal mass blowup solution to (1.1), $\mu = \pm 1$, with $N(t) \leq 1$ and*

$$\int_0^\infty N(t)^3 dt = \tilde{K} < \infty.$$

Then

$$\|u(t,x)\|_{L_t^\infty \dot{H}_x^2([0,\infty)\times\mathbf{R})} \lesssim_{m_0} \tilde{K}^2. \tag{6.1}$$

By (2.40) there exists a uniform K_0 such that if M is any dyadic integer and $[0,T]$ is a compact interval with

$$\int_0^T \int |u(t,x)|^6 dx dt = M\epsilon_0^6, \tag{6.2}$$

$$\sum_{J_l \subset [0,T]} N(J_l) = \delta K \leq \delta K_0.$$

After rescaling, $u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x)$, $\lambda = \frac{M}{K}$, by theorem 4.2,

$$\|u_\lambda\|_{\tilde{X}_M([0,\frac{T}{\lambda^2}]\times\mathbf{R})} \leq C, \tag{6.3}$$

with C independent of T. For $l \geq 5$ let

$$\mathcal{U}(2^l) = \sup_T \|P_{>2^l K_0} u\|_{U_\Delta^2([0,T]\times\mathbf{R})}. \tag{6.4}$$

By Duhamel's formula

$$\|P_{>2^l K_0} u\|_{U_\Delta^2([0,T]\times\mathbf{R})} \lesssim_{m_0} \|P_{>2^l K_0} u(T)\|_{L_x^2(\mathbf{R})}$$

$$+\|P_{>2^l K_0}(|u|^4 u)\|_{DU_\Delta^2([0,T]\times\mathbf{R})}.$$

Take $l \geq 5$. By theorem 5.5,

$$\| \int_0^t e^{i(t-\tau)\Delta} (P_{>K_0} u)(P_{\leq 2^{-10} K_0} u)^4(\tau) d\tau \|_{U_\Delta^2([0,T] \times \mathbf{R})} \lesssim \sum_{K_0 \leq N_2} \frac{K_0}{N_2} \lesssim 1.$$

Splitting the Duhamel term,

$$\| | |P_{>K_0} u| |P_{\geq 2^{-10} K_0} u| |u|^3 \|_{N^0([0,T] \times \mathbf{R})} \lesssim \| |P_{>K_0} u| |P_{\geq 2^{-10} K_0} u|^4 \|_{L_{t,x}^{6/5}([0,T] \times \mathbf{R})}$$

$$+ \| |P_{>K_0} u| |P_{\geq 2^{-10} K_0} u| |P_{\leq 2^{-10} K_0} u|^3 \|_{L_t^{4/3} L_x^1([0,T] \times \mathbf{R})}.$$

$$\| |P_{>K_0} u| |P_{\geq 2^{-10} K_0} u|^4 \|_{L_{t,x}^{6/5}([0,T] \times \mathbf{R})} \lesssim \| P_{>K_0} u \|_{L_{t,x}^6([0,T] \times \mathbf{R})} \| P_{\geq 2^{-10} K_0} u \|_{L_{t,x}^6([0,T] \times \mathbf{R})}^4 \lesssim 1.$$

We use

$$\| P_{\geq 2^{-10} K_0} u \|_{L_{t,x}^6([0,T] \times \mathbf{R})} \lesssim \| u_\lambda \|_{\tilde{X}_M([0, \frac{T}{\lambda^2}] \times \mathbf{R})}$$

along with Littlewood-Paley summation and the definition of the $\tilde{X}_M$ seminorm. By theorem 5.1,

$$\| |P_{>K_0} u| |P_{\leq 2^{-10} K_0} u|^3 |P_{>2^{-10} K_0} u| \|_{L_t^{4/3} L_x^1([0,T] \times \mathbf{R})}$$

$$\lesssim \| |P_{>K_0} u| |P_{\leq 2^{-10} K_0} u|^2 \|_{L_{t,x}^2([0,T] \times \mathbf{R})}^{3/2} \| P_{>2^{-10} K_0} u \|_{L_t^\infty L_x^2([0,T] \times \mathbf{R})}^{1/2} \| P_{\leq 2^{-10} K_0} u \|_{L_t^\infty L_x^2([0,T] \times \mathbf{R})}^{1/2}$$

$$+ \| |P_{>K_0} u| |P_{\leq 2^{-10} K_0} u|^2 \|_{L_{t,x}^2([0,T] \times \mathbf{R})} \| P_{2^{-10} K_0 \leq \cdot \leq K_0} u \|_{L_t^4 L_x^\infty([0,T] \times \mathbf{R})} \| u \|_{L_t^\infty L_x^2([0,T] \times \mathbf{R})} \lesssim 1.$$

Therefore, $\mathcal{U}(2^l) \lesssim 1$ when $l \geq 5$. Because $\sum_{J_l \subset [0,T]} N(J_l) \leq \delta K_0$ for any T, $|\xi(t) - \xi(0)| \leq 2^{-20} K_0$ for all $t \in [0, \infty)$.

$$\sum_{J_l \subset [0,T]} N(J_l) \leq \delta K_0$$

also implies $\lim_{t \to \infty} N(t) = 0$, which implies $\lim_{t \to \pm\infty} \| P_{2^l K_0} u(t) \|_{L_x^2(\mathbf{R})} = 0$ for $l \geq L_0$ for some fixed L_0. Therefore,

$$\sup_T \| P_{>2^l K_0} u \|_{U_\Delta^2([0,T] \times \mathbf{R})} \lesssim \sup_T \| P_{>2^l K_0} (|u|^4 u) \|_{DU_\Delta^2([0,T] \times \mathbf{R})}. \tag{6.5}$$

By theorem 5.5,

$$\| P_{>2^l K_0} ((P_{>2^{l-5} K_0} u)(P_{\leq 2^{-10} K_0} u)^4) \|_{DU_\Delta^2([0,T] \times \mathbf{R})} \lesssim 2^{-l/2} \| P_{>2^{l-5} K_0} u \|_{U_\Delta^2([0,T] \times \mathbf{R})}. \tag{6.6}$$

By theorem 5.4,

$$\|P_{>2^l K_0}((P_{>2^{l-5}K_0}u)(P_{>2^{-10}K_0}u)u^3)\|_{DU^2_\Delta([0,T]\times\mathbf{R})}$$
$$\lesssim \|P_{>2^{l-5}K_0}u\|_{U^2_\Delta([0,T]\times\mathbf{R})}(\sum_{j=0}^{l-5}\frac{2^{j/4}}{2^{l/4}}\|P_{>2^j K_0}u\|^{1/2}_{L_t^\infty L_x^2([0,\infty)\times\mathbf{R})}). \tag{6.7}$$

Because

$$\sup_T \|P_{>2^j K_0}u\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R})} \to 0 \tag{6.8}$$

as $j\to\infty$, there exists L_0 such that for $l \geq L_0$,

$$\sup_T \|P_{>2^l K_0}u\|_{U^2_\Delta([0,T]\times\mathbf{R})} \leq 2^{-15}\sup_T \|P_{>2^{l-5}K_0}u\|_{U^2_\Delta([0,T]\times\mathbf{R})}. \tag{6.9}$$

Therefore,

$$\sup_T \|P_{>2^l K_0}u\|_{U^2_\Delta([0,T]\times\mathbf{R})} \lesssim_{m_0} 2^{-3l}$$

for $l \geq L_0$, which proves $u(t)\in L_t^\infty \dot{H}_x^2([0,\infty)\times\mathbf{R})$, and

$$\|u(t,x)\|_{L_t^\infty \dot{H}_x^2([0,\infty)\times\mathbf{R})} \lesssim_{m_0} K_0^2. \tag{6.10}$$

□

Take some $\eta(t)\to 0$, possibly very slowly.

$$\|e^{-ix\cdot\xi(t)}u(t)\|_{\dot{H}^1(\mathbf{R})} \lesssim N(t)C(\eta(t)) + \eta(t)^{1/2}. \tag{6.11}$$

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \frac{1}{6}\int |u(t,x)|^6 dx. \tag{6.12}$$

By energy conservation $E(u(t)) = E(u(0))$ for any t.

Now, by Holder's inequality,

$$\int |u(0,x)|^2 dx \leq \int_{|x-x(0)|\leq \frac{C(\frac{m_0^2}{1000})}{N(0)}} |u(0,x)|^2 dx + \frac{m_0^2}{1000}$$
$$\leq C\|u\|^2_{L_x^6(\mathbf{R})}\frac{C(\frac{m_0^2}{1000})^{2/3}}{N(0)^{2/3}} + \frac{m_0^2}{1000}$$
$$\leq CE(u(0))^{1/3}\frac{C(\frac{m_0^2}{1000})^{2/3}}{N(0)^{2/3}} + \frac{m_0^2}{1000}.$$

Now by (6.11), mass conservation, and the Sobolev embedding theorem, we can choose t sufficiently large so that after a Galilean transformation setting $\xi(t) = 0$,

$$CE(u(t))^{1/2} \frac{C(\frac{m_0^2}{1000})}{N(0)} + \frac{m_0^2}{1000} \leq \frac{m_0^2}{100}.$$

But since $E(u(0)) = E(u(t))$, this implies $\int |u(0,x)|^2 dx \leq \frac{m_0^2}{100}$, which contradicts mass conservation. This completes the proof of theorem 6.1. $\Box$

Remark: We cannot apply these arguments exactly to the focusing case because E is no longer positive definite when $\mu = -1$. These arguments do apply when $\mu = -1$ and $\|u_0\|_{L^2(\mathbf{R})}$ is less than the mass of the ground state. We will not discuss this matter here.

7 The case $\int_0^\infty N(t)^3 dt = \infty$

As in the cases when $d \geq 3$, $d = 2$, we defeat this scenario via a frequency localized Morawetz estimate. [7] proved that in the defocusing case

$$\|u(t,x)\|_{L_{t,x}^8([0,T]\times\mathbf{R})}^8 \lesssim \|u(t)\|_{L_t^\infty \dot{H}^1([0,T]\times\mathbf{R})} \|u(t)\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R})}^3. \tag{7.1}$$

See also [27]. The interaction Morawetz estimate is not positive definite in the focusing case. Let $\chi \in C_0^\infty(\mathbf{R})$ be an even function,

$$\chi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| > 2. \end{cases} \tag{7.2}$$

Here we prove

Theorem 7.1 *Suppose $u(t,x)$ is a minimal mass blowup solution to (1.1), $\mu = +1$, on $[0,T]$ with $N(t) \leq 1$,*

$$\int_0^T \int |u(t,x)|^6 dx dt = M\epsilon_0^6 \tag{7.3}$$

for some dyadic integer M and for $\|u\|_{L_{t,x}^6(J_l\times\mathbf{R})} = \epsilon_0$,

$$\sum_{J_l \subset [0,T]} N(J_l) = \delta K. \tag{7.4}$$

Take $\lambda = \frac{M}{K}$. Let

$$\widehat{Iu}(t,\xi) = \chi(\frac{\xi}{32M}) \hat{u}_\lambda(t,\xi). \tag{7.5}$$

Then

$$\|Iu_\lambda\|_{L_{t,x}^8([0,T]\times\mathbf{R})}^8 \lesssim o(K)(\frac{M}{K}), \tag{7.6}$$

$M^I(t)$ is a modification of the Morawetz action in [7] (see (7.10)).

Proof: Since we are going to work exclusively with the rescaled function u_λ, we will drop the λ in our notation and realize that we are working with u_λ for the rest of this section. [7] defined the action

$$M(t) = \frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}} a(x-y)|u(t,y)|^2 Im[\bar{u}(t,x)\partial_x u(t,x)]dxdy, \tag{7.7}$$

$$a(x-y) = erf(\frac{x-y}{\epsilon}) = \int_{-\infty}^{\frac{x-y}{\epsilon}} e^{-t^2}dt. \tag{7.8}$$

Taking the limit $\epsilon \to 0$,

$$\int_0^T \int |u(t,x)|^8 dxdt \lesssim \int_0^T \partial_t M(t) \lesssim \sup_{[0,T]} |M(t)|. \tag{7.9}$$

Because of conservation of mass and momentum

$$\frac{\partial}{\partial t}\int\int |u(t,y)|^2 Im[\bar{u}(t,x)\partial_x u(t,x)]dxdy = 0,$$

therefore

$$a(x-y) = \int_0^{\frac{x-y}{\epsilon}} e^{-t^2}dt$$

gives exactly the same Morawetz estimates. We will use this $a(x-y)$ because it is an odd function of $x-y$. Now define the modified action

$$M_I(t) = \frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}} a(x-y)|Iu(t,y)|^2 Im[\bar{Iu}(t,x)\partial_x Iu(t,x)]dxdy. \tag{7.10}$$

We have

$$\partial_t(Iu) = i\Delta(Iu) - i|Iu|^4(Iu) + i|Iu|^4(Iu) - iI(|u|^4 u). \tag{7.11}$$

If we simply had

$$\partial_t(Iu) = i\Delta(Iu) - i|Iu|^4(Iu),$$

then we would have

$$\int_0^T \int |Iu(t,x)|^8 dxdt \lesssim \int_0^T \partial_t M(t) \lesssim \sup_{[0,T]} |M_I(t)|, \tag{7.12}$$

following the arguments in [6] identically. Instead we have

$$\int_0^T \int |Iu(t,x)|^8 dxdt \lesssim \int_0^T \partial_t M_I(t) + \mathcal{E} \lesssim \sup_{[0,T]} |M(t)| + \mathcal{E}, \tag{7.13}$$

where

$$\begin{aligned}\mathcal{E} = \frac{1}{4}\int_0^T \int\int a(x-y)[I(|u|^4\bar{u})(t,y)Iu(t,y) - I(|u|^4 u)(t,y)\overline{Iu}(t,y)] \\ \times[\overline{Iu}(t,x)\partial_x Iu(t,x) - Iu(t,x)\partial_x \overline{Iu}(t,x)]dxdydt\end{aligned} \tag{7.14}$$

$$\begin{aligned}+\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[(|Iu|^4(Iu)(t,x) - I(|u|^4u)(t,x))(\partial_x\overline{Iu}(t,x)) \\ +(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))(\partial_x Iu(t,x))dxdydt.\end{aligned} \tag{7.15}$$

$$\begin{aligned}+\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[\overline{Iu}(t,x)\partial_x(|Iu|^4(Iu)(t,x) - I(|u|^4u)(t,x)) \\ +Iu(t,x)\partial_x(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))]dxdydt.\end{aligned} \tag{7.16}$$

The interaction Morawetz estimates are Galilean invariant. Indeed, because $a(x-y)$ is an odd function,

$$\int\int a(x-y)|Iu(t,y)|^2 Im[i\xi(t)|Iu(t,x)|^2]dxdy \equiv 0. \tag{7.17}$$

Therefore,

$$M_I(t) = \int\int a(x-y)|Iu(t,y)|^2 Im[\overline{Iu}(t,x)(\partial_x - i\xi(t))Iu(t,x)]dxdy. \tag{7.18}$$

Also,

$$\frac{1}{4}\int_0^T \int\int a(x-y)[I(|u|^4\bar{u})(t,y)Iu(t,y) - I(|u|^4u)(t,y)\overline{Iu}(t,y)](2i\xi(t))|Iu(t,x)|^2 dxdydt$$

$$\begin{aligned}+\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[(|Iu|^4(Iu)(t,x) - I(|u|^4u)(t,x))((-i\xi(t))\overline{Iu}(t,x)) \\ +(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))((i\xi(t))Iu(t,x))dxdydt\end{aligned}$$

$$+\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[\overline{Iu}(t,x)(i\xi(t))(|Iu|^4(Iu)(t,x) - I(|u|^4u)(t,x))$$
$$+Iu(t,x)(-i\xi(t))(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))]dxdydt \equiv 0.$$

Therefore,

$$\begin{aligned}\mathcal{E} = \frac{1}{4}\int_0^T \int\int a(x-y)[I(|u|^4\bar{u})(t,y)Iu(t,y) - I(|u|^4u)(t,y)\overline{Iu}(t,y)] \\ \times[\overline{Iu}(t,x)(\partial_x - i\xi(t))Iu(t,x) - Iu(t,x)(\partial_x + i\xi(t))\overline{Iu}(t,x)]dxdydt\end{aligned} \tag{7.19}$$

$$\begin{aligned}+\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[(|Iu|^4(Iu)(t,x) - I(|u|^4u)(t,x))((\partial_x + i\xi(t))\overline{Iu}(t,x)) \\ +(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))((\partial_x - i\xi(t))Iu(t,x))dxdydt\end{aligned} \tag{7.20}$$

$$\begin{aligned}+\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[\overline{Iu}(t,x)(\partial_x - i\xi(t))(|Iu|^4(Iu)(t,x) - I(|u|^4u)(t,x)) \\ +Iu(t,x)(\partial_x + i\xi(t))(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))]dxdydt.\end{aligned} \tag{7.21}$$

Let $u_l = P_{\leq \frac{M}{32}}u$ and $u_l + u_h = u$.

$$|u_h|^2|u|^4 \lesssim |u_h|^2|u_{\leq 2^{-10}M}|^4 + |u_h|^2|u_{\geq 2^{-10}M}|^4.$$

By theorem 5.1, corollary 5.2,

$$\||u_h|^2|u_{\leq 2^{-10}M}|^4\|_{L^1_{t,x}([0,T]\times\mathbf{R})} \lesssim (\sup_{J_l}\|P_{\frac{M}{32}}u\|_{U^2_\Delta(J_l\times\mathbf{R})})^2 C_0 + \|P_{>N(t)C_0}u\|^2_{L^\infty_t L^2_x([0,T]\times\mathbf{R})}. \tag{7.22}$$

By Duhamel's formula, $\|u\|_{L^4_t L^\infty_x(J_l\times\mathbf{R})} \lesssim_{m_0} 1$, and $N(t) \leq \frac{M}{K}$ on $[0,T]$,

$$\|P_{>\frac{M}{32}}u\|_{U^2_\Delta(J_l\times\mathbf{R})} \lesssim \|P_{>\frac{M}{32}}u\|_{L^\infty_t L^2_x([0,T]\times\mathbf{R})}$$
$$+\|P_{>2^{-10}M}u\|_{L^\infty_t L^2_x([0,T]\times\mathbf{R})}\|u\|_{L^4_t L^\infty_x([0,T]\times\mathbf{R})} \leq o(1),$$

with $o(1) \to 0$ as $K \to \infty$. Let

$$C_0 = (\sup \|P_{>2^{-10}M}u\|_{U^2_\Delta(J_l\times\mathbf{R})})^{-1}, \tag{7.23}$$

$C_0 \nearrow \infty$ as $K \to \infty$, so

$$(\sup_{J_l} \|P_{>2^{-10}M} u\|_{U^2_\Delta(J_l \times \mathbf{R})})^2 C_0 + \|P_{\xi(t), \geq C_0 N(t)} u\|^2_{L_t^\infty L_x^2([-T,T] \times \mathbf{R})} \leq o(1).$$

$$\| |u_h|^2 |u_{\geq 2^{-10}M}|^4 \|_{L^1_{t,x}([0,T] \times \mathbf{R})}$$

$$\lesssim \|u_h\|^2_{L_t^5 L_x^{10}([0,T] \times \mathbf{R})} \|u_{\geq 2^{-10}M}\|^3_{L_t^5 L_x^{10}([0,T] \times \mathbf{R})} \|u_{\geq 2^{-10}M}\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R})} \leq o(1).$$

Now we are ready to estimate

$$|M_I(t)| + |(7.19)| + |(7.20)| + |(7.21)|.$$

We start with $|M_I(t)|$. Because $N(t) \leq \frac{M}{K}$,

$$|M_I(t)| \lesssim \|u\|^3_{L_t^\infty L_x^2([-T,T] \times \mathbf{R})} \|(\partial_x - i\xi(t)) I u\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R})} \lesssim o(K)(\frac{M}{K}). \tag{7.24}$$

Next we take (7.20). Because $I = 1$ on $|\xi| \leq 32M$, u_l is supported on $2^{-5}M$,

$$|Iu_l|^4 (Iu_l) - I(|u_l|^4 u_l) \equiv 0.$$

Because $(\partial_x - i\xi(t)) I \lesssim M$,

$$(7.20) = \frac{5}{2} \int_{-T}^{T} \int \int a(x-y) |Iu(t,y)|^2 Re[[u_l^4 I u_h - I(u_l^4 u_h)](\partial_x - i\xi(t))(Iu)](t,x) dx dy dt$$

$$+ M \| |u_h|^2 |u|^4 \|_{L^1_{t,x}([-T,T] \times \mathbf{R})} \|Iu\|^2_{L_t^\infty L_x^2([-T,T] \times \mathbf{R})}.$$

Also, it suffices for us to consider only $P_{\geq 8M} u$ since we will have cancellation otherwise. Make a Littlewood - Paley decomposition. By the fundamental theorem of calculus,

$$|m(\xi + \xi_2) - m(\xi)| \leq |\xi_2| \sup |\partial_x m(\xi)|.$$

$$\| |u_l|^4 (IP_{>\frac{M}{4}} u) - I(u_l^4 (P_{>\frac{M}{4}} u)) \|_{L_t^{6/5} L_x^{6/5}([0,T] \times \mathbf{R})}$$

$$\lesssim \sum_{N_5 \leq N_4 \leq N_3 \leq N_2 \leq \frac{M}{32}} (\frac{N_2}{M}) \|(u_h)(P_{N_2} u)\|_{L^2_{t,x}([0,T] \times \mathbf{R})} \|P_{N_3} u\|_{L_t^4 L_x^\infty([0,T] \times \mathbf{R})}$$

$$\times \|P_{N_4} u\|_{L_t^{12} L_x^3([0,T] \times \mathbf{R})} \|P_{N_5} u\|_{L^\infty_{t,x}([0,T] \times \mathbf{R})}$$

$$\lesssim \sum_{N_5 \leq N_4 \leq N_3 \leq N_2 \leq 2^{-10}M} (\frac{N_2}{M})(\frac{M}{N_2})^{1/2} (\frac{M}{N_3})^{1/4} (\frac{M}{N_4})^{1/12} (\frac{N_5}{M})^{1/2} \lesssim 1.$$

The second to last inequality follows from lemma 4.1 and $\|u\|_{\tilde{X}_M} \leq C$. Meanwhile,

$$\|(\partial_x - i\xi(t))Iu\|_{L^6_{t,x}([0,T]\times\mathbf{R})} \lesssim \sum_{N\leq M} N(\frac{M}{N})^{1/6} o(1) \lesssim o(K)(\frac{M}{K}).$$

Therefore, $|(7.20)| \lesssim o(K)(\frac{M}{K})$.

Next, integrating by parts,

$$\int_0^T \int\int a(x-y)|Iu(t,y)|^2[\overline{Iu}(t,x)](\partial_x - i\xi(t))[|Iu|^4(Iu) - I(|u|^4u)](t,x)dxdydt$$

$$= -\int_0^T \int\int a(x-y)|Iu(t,y)|^2[(\partial_x + i\xi(t))\overline{Iu}(t,x)][|Iu|^4(Iu) - I(|u|^4u)](t,x)dxdydt$$

$$-\int_0^T \int\int \partial_x a(x-y)\overline{Iu}(t,x)[|Iu|^4(Iu) - I(|u|^4u)](t,x)|Iu(t,y)|^2 dxdydt.$$

By Young's inequality, since $\|\partial_x a(x-y)\|_{L^1_x(\mathbf{R})} = 1$,

$$\int_0^T \int\int \partial_x a(x-y)\overline{Iu}(t,x)[|Iu|^4(Iu) - I(|u|^4u)](t,x)|Iu(t,y)|^2 dxdydt$$

$$\lesssim \|I(u_l^4 u_h) - u_l^4(Iu_h)\|_{L^{6/5}_{t,x}([0,T]\times\mathbf{R})}\|Iu\|_{L^{12}_t L^{18}_x([0,T]\times\mathbf{R})}$$

$$+\|u_h\|^2_{L^4_t L^\infty_x([0,T]\times\mathbf{R})}\|Iu\|^6_{L^{12}_t L^6_x([0,T]\times\mathbf{R})}$$

$$+\|u_h\|^5_{L^5_t L^{10}_x([0,T]\times\mathbf{R})}\|Iu\|^3_{L^\infty_t L^2_x([0,T]\times\mathbf{R})} \leq o(K)(\frac{M}{K}).$$

Therefore, $(7.21) = (7.20) + o(K)(\frac{M}{K})$, so $(7.21) \leq o(K)(\frac{M}{K})$.

Finally we turn to (7.19).

$$I(|u|^4u)\overline{Iu} - I(|u|^4\bar{u})(Iu) = [I(|u|^4u) - |Iu|^4(Iu)]\overline{Iu} + [|Iu|^4(\overline{Iu}) - I(|u|^4\bar{u})]Iu.$$

$$\int_0^T \int\int |Iu(t,x)||(\partial_x - i\xi(t))Iu(t,x)||u_h(t,y)|^2|u(t,y)|^4 dxdydt$$

$$\lesssim M\|Iu(t,x)\|^2_{L^\infty_t L^2_x([0,T]\times\mathbf{R})}\||u_h(t,y)|^2|u(t,y)|^4\|_{L^1_{t,x}([0,T]\times\mathbf{R})} \lesssim o(K)(\frac{M}{K}).$$

Finally, since

$$I(u_l^5) - (Iu_l)^5 \equiv 0,$$

it remains to evaluate

$$\int_0^T \int \int |Iu(t,x)||(\partial_x - i\xi(t))Iu(t,x)|u_l(t,y)^5 (P_{\geq M} u(t,y)) a(x-y) dx dy dt$$

$$= \int_0^T \int \int a(x-y)|Iu(t,x)||(\partial_x - i\xi(t))Iu(t,x)| \frac{\Delta}{\Delta}[u_l(t,y)^5 (P_{\geq M} u(t,y))] dx dy dt$$

Integrating by parts

$$\lesssim \int_0^T \int \int |Iu(t,x)||(\partial_x - i\xi(t))Iu(t,x)|(\partial_x a(x-y)) \frac{1}{M} u_l(t,y)^5 u_h(t,x) dx dy dt.$$

Again by Young's inequality,

$$\lesssim \|u_h(t,y)\|_{L_t^4 L_x^\infty([0,T]\times \mathbf{R})} \|(\partial_x - i\xi(t))Iu(t,x)\|_{L_t^\infty L_x^2([0,T]\times \mathbf{R})} \|Iu(t,x)\|_{L_t^{12} L_x^6([0,T]\times \mathbf{R})}^6 \lesssim o(K)(\frac{M}{K}).$$

This completes the proof of theorem 7.1. $\Box$

Remark: The only properties of $a(x-y)$ that we used in the estimate of (7.19), (7.20), and (7.21) are a is an odd function and there exists a constant C such that

$$|a(x)| \leq C, \tag{7.25}$$

and

$$\|\partial_x a(x)\|_{L^1(\mathbf{R})} \leq C. \tag{7.26}$$

Therefore, we have in fact proved

Theorem 7.2 *Suppose $a(t,x)$ is an odd function of x for all t,*

$$|a(t,x)| \leq C, \tag{7.27}$$

$$\|\partial_x a(t,x)\|_{L^1(\mathbf{R})} \leq C. \tag{7.28}$$

Then if $u(t,x)$ is a minimal mass blowup solution to (1.1), $\mu = \pm 1$,

$$\frac{1}{4}\int_0^T \int\int a(t,x-y)[I(|u|^4\bar{u})(t,y)Iu(t,y) - I(|u|^4 u)(t,y)\overline{Iu}(t,y)]$$
$$\times[\overline{Iu}(t,x)(\partial_x - i\xi(t))Iu(t,x) - Iu(t,x)(\partial_x + i\xi(t))\overline{Iu}(t,x)]dxdydt \lesssim_{m_0,d} o(K)C, \tag{7.29}$$

$$\frac{1}{4}\int_0^T \int\int a(t,x-y)|Iu(t,y)|^2[(|Iu|^4(Iu)(t,x) - I(|u|^4 u)(t,x))((\partial_x + i\xi(t))\overline{Iu}(t,x))$$
$$+(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))((\partial_x - i\xi(t))Iu(t,x))dxdydt \lesssim_{m_0,d} o(K)C, \tag{7.30}$$

and

$$\frac{1}{4}\int_0^T \int\int a(x-y)|Iu(t,y)|^2[\overline{Iu}(t,x)(\partial_x - i\xi(t))(|Iu|^4(Iu)(t,x) - I(|u|^4 u)(t,x))$$
$$+Iu(t,x)(\partial_x + i\xi(t))(|Iu|^4(\overline{Iu})(t,x) - I(|u|^4\bar{u})(t,x))]dxdydt \lesssim_{m_0,d} o(K)C. \tag{7.31}$$

Remark: We will not use the interaction Morawetz estimate of [7], [27] for the focusing problem because the interaction Morawetz estimate is not positive definite when $\mu = -1$. Nevertheless, if we did have an interaction Morawetz estimate, theorem 7.2 implies that the Fourier truncation error is bounded by $o(K)C$ if a satisfies (7.27), (7.28).

Theorem 7.3 *There does not exist a minimal mass blowup solution with $N(t) \leq 1$, $\int_0^\infty N(t)^3 dt = \infty$.*

Proof: Suppose there did exist a minimal mass blowup solution with $N(t) \leq 1$ and $\int_0^\infty N(t)^3 dt = \infty$. Take a compact time interval $[0,T]$ with

$$\int_0^T \int |u(t,x)|^6 dxdt = M\epsilon_0^6,$$

M a dyadic integer. $[0,T]$ can be partitioned into M small intervals with $\|u(t,x)\|_{L^6_{t,x}(J_l \times \mathbf{R})} = \epsilon_0$. We have

$$\sum_{J_l \subset [0,T]} N(J_l) = \delta K.$$

Rescaling, $u(t,x) \mapsto \lambda^{1/2} u(\lambda^2 t, \lambda x)$, let $\lambda = \frac{M}{K}$. Let $u_\lambda(t,x)$ be the rescaled solution. $[0, \frac{T}{\lambda^2}]$ can be partitioned into M small intervals J_l^λ, and

$$\sum_{J_l^\lambda \subset [0,\frac{T}{\lambda^2}]} N(J_l^\lambda) = \delta M.$$

Since $|\xi(t)| \leq 2^{-20} M$ for $t \in [0, T]$, and

$$\int_{|\xi - \xi(t)| > C(\frac{m_0^2}{1000}) N(t)} |\hat{u}(t, \xi)|^2 d\xi, \tag{7.32}$$

for K sufficiently large,

$$\frac{m_0^2}{2} \leq \int_{|x - x(t)| \leq \frac{C(\frac{m_0^2}{1000})}{N(t)}} |Iu(t, x)|^2 dx. \tag{7.33}$$

Therefore,

$$\int_0^T N(t)^3 \frac{m_0^8}{16} dt \leq \int_0^T N(t)^3 (\int_{|x - x(t)| \leq \frac{C(\frac{m_0^2}{1000})}{N(t)}} |Iu(t, x)|^2 dx)^4 dt. \tag{7.34}$$

By Holder's inequality, and theorem 7.1,

$$(7.34) \lesssim \int_0^T N(t)^3 (\frac{C(\frac{m_0^2}{1000})}{N(t)})^3 \|Iu(t)\|_{L_x^8(\mathbf{R})}^8 dt \lesssim o(K) \frac{M}{K}. \tag{7.35}$$

Since $(7.34) \sim M$, the proof of theorem 7.3 is complete. $\Box$

This completes the proof of theorem 1.6.

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